Journal of Statistical Physics, Vol. 82, Nos. 3/4, 1996

Effective Potential and Weak Noise Transitions

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Received April 11, 1995; final June 5, 1995

We review the notion of effective potential for stochastic processes and discuss its possible applications. We calculate this function up to first order in a parameter measuring the intensity of the noise for a general nonlinear system. The result is applied exhibiting a transition induced by weak noise.

KEY WORDS: Effective potential; noise-induced transitions; weak noise transitions; stochastic processes.

1. INTRODUCTION

In many realistic situations macroscopic systems can be modeled by Markov processes for some macrovariables. This description can be through a master equation or through stochastic differential equations. Here we shall limit ourselves to the second description, but our methods can be generalized directly to master equations, as we have already discussed in some simple cases.[1] Furthermore, we shall not consider inhomogeneous fluctuations in space. Our discussion will be limited to equations for n macrovariables $\mathbf{q}=(q_1 ..., q_n)$ of the form (sum over repeated indices is to be understood from now on)

$$\dot{q}_\mu(t) = B_\mu(\mathbf{q}(t)) + \sqrt{\eta}\, \sigma^\mu_j(\mathbf{q}(t))\, \xi^j(t) \tag{1}$$

where $(\xi^1(t),..., \xi^k(t))$ is a set of Gaussian white noises with zero mean value and δ-correlated with correlations $\langle \xi^j(t)\, \xi^k(t')\rangle = c^{jk}\delta(t-t')$ and η is a parameter measuring the intensity of the noise. Although ordinary stochastic differential equations, in contrast to partial differential equations, have a unique stationary probability and a unique mean value for

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the macrovariables, one can have metastable states represented by maxima of the stationary probability. The problem of the appearance of these local maxima, their number and location, and how they change when the parameters of the system change is what we want to consider here. In general the stationary probability is not accessible and no systematic method of calculation is available. In the case of weak noise [small η in (1)] the method of the nonequilibrium potential pioneered by Graham and collaborators[2] can be used, but it can present some difficulties especially in the case when one has more than one maximum[3,4] and this in spite of recent successes of the method in local expansions for extended systems.[5,6]

We shall explore here the possibility of using another function, the effective potential, which has the property of being minimal for the unique mean value. However, if the effective potential is calculated as a power series in η, which is always possible, it presents minima for the mean values of metastable states (in this case the stationary probability has several local maxima). The use of this function was proposed by Graham[7] and recently it has been used by Bonilla.[8] Let us explain precisely what we understand by a metastable state. This notion is associated with local maxima of the stationary probability $p_{st}(\mathbf{q})$ and with each maximum we associate a metastable state [we shall consider here only the case where $p_{st}(\mathbf{q})$ is maximal at points and not in curves or surfaces]. Suppose $p_{st}(\mathbf{q})$ has two local maxima at the points $(\mathbf{b}'_1, \mathbf{b}_2)$ with escape times τ_1 and τ_2,[9] then a realization of the stochastic process which starts near enough to $\mathbf{b}'_1$ at time zero fluctuate around a mean value $\mathbf{b}_1$ for a time t of the order of τ_1 and then jump to the second metastable state and fluctuate around a mean value $\mathbf{b}_2$ for a time of the order of τ_2 and then come back to the state, starting again as at time zero. The quantities $\delta\mathbf{b}_j = \mathbf{b}'_j - \mathbf{b}_j$ will be of the order of the fluctuations and we shall have a similar situation if we have more than two local maxima. The perturbative effective potential is a function $U(\mathbf{q})$ represented by a series in η of the form

$$U(\mathbf{q}) = U_0(\mathbf{q}) + \eta U_1(\mathbf{q}) + \eta^2 U_2(\mathbf{q}) + \cdots \tag{2}$$

in which every term $U_\alpha(\mathbf{q})$, $\alpha \geqslant 0$, can be calculated in a systematic way which corresponds in fact, as we shall see in Section 2, to expanding first in η and then taking the limit $t_0 \to -\infty$, where t_0 is the time at which we impose an initial condition when we solve (1). This function $U(\mathbf{q})$ has the property that the n equations

$$\frac{\partial U(\mathbf{q})}{\partial q_\alpha} \equiv G_\alpha(\mathbf{q}) = 0, \qquad \alpha = 1,..., n \tag{3}$$

determine the mean values $(\mathbf{b}_1,\ldots,\mathbf{b}_n)$ in the possible metastable states in the following sense: if $G_\alpha(\mathbf{b}_k)=0$, $k=1,\ldots,m$, and $U(\mathbf{b}_k)$ is a local minimum, then $\mathbf{b}_k$ is a mean value. Moreover, each $\mathbf{b}_k$ is in correspondence with a $\mathbf{b}'_k$ such that $p_{\mathrm{st}}(\mathbf{b}'_k)$ is a local maximum.

These properties are direct consequences of the definition of $U(\mathbf{q})$. Let $\hat{Z}(\mathbf{J};\tau)$ be the generating functional for a constant time-independent source $\mathbf{J}$ of the process (1) with initial condition $\mathbf{q}(t_0)=\boldsymbol{\alpha}_0$ defined by

$$\hat{Z}(\mathbf{J};\tau)=\left\langle \exp\left(\frac{J_\mu}{\eta}\int_{t_0}^{T} dt\, q_\mu(t)\right)\right\rangle_{(\tau,\boldsymbol{\alpha}_0)} \tag{4}$$

where $\tau=T-t_0$ and $\hat{Z}(\mathbf{0};\tau)=1$. We write $\hat{Z}(\mathbf{J};\tau)=\exp[(1/\eta)\,w(\mathbf{J},\tau)]$ and we assume that the following limits exists:

$$\lim_{\tau\to\infty}\frac{\eta}{\tau}\ln\hat{Z}(\mathbf{J};\tau)=\bar{w}(\mathbf{J}) \tag{5}$$

In perturbation theory it exists at each order and then one has that $w(\mathbf{J},\tau)\to\tau\bar{w}(\mathbf{J})$, when $\tau\to\infty$, and the limit is independent of $\boldsymbol{\alpha}_0$ in a sense which will be discussed later. The function $U(\mathbf{b}_k)$ is defined as the Legendre transform of $\bar{w}(\mathbf{J})$ by the formulas $(\mathbf{J}\cdot\mathbf{q}=J_\mu q_\mu)$

$$U(\mathbf{q})+\bar{w}(\mathbf{J})=\mathbf{J}\cdot\mathbf{q} \tag{6}$$

$$\frac{\partial\bar{w}(\mathbf{J})}{\partial J_\mu}=q_\mu,\qquad \frac{\partial U(\mathbf{q})}{\partial q_\mu}=J_\mu \tag{7}$$

In order zero in η we shall see that for each local attractor of the deterministic system $\dot{q}_\mu=B_\mu(\mathbf{q})$ associated with (1) we can invert the first equation in (7) to express J_μ as a function of $\mathbf{q}$ for $\mathbf{J}$ in a region $\mathscr{D}_1$ around $\mathbf{J}=0$ to which will correspond a region $\mathscr{D}_2$ of variation of $\mathbf{q}$ containing the attractor. Once this is done there is no more problem with the inversion of (7) in higher orders and the important point is that the analytic form of the function $U(\mathbf{q})$ calculated in this way will be independent of the initial attractor. This procedure will generate a function $U(\mathbf{q})$ which will be convex in each of the regions $\mathscr{D}_2$. From (7) and the definition of $\hat{Z}(\mathbf{J};\tau)$ we see that at $\mathbf{J}\doteq\mathbf{0}$ the derivative $\partial\bar{w}(\mathbf{J})/\partial J_\alpha$ takes the value

$$\left.\frac{\partial\bar{w}(\mathbf{J})}{\partial J_\alpha}\right|_{\mathbf{J}=\mathbf{0}}=\left\langle\frac{1}{\tau}\int_{t_0}^{T}dt\,q_\alpha(t)\right\rangle_{\tau\to\infty}=\bar{q}_\alpha \tag{8}$$

since $\langle q^\alpha(t)\rangle_{\tau\to\infty}=\bar{q}_\alpha$ is the time-independent mean value in the stationary state of the process (1). In fact $\bar{q}_\alpha$ will be the component α of one of the mean values $\mathbf{b}_k$ discussed above. The reciprocal relation of (8), i.e., the second equation in (7), is then the fundamental property (3) of the effective potential. The property that $\{\bar{q}_\alpha\}$ is a minimum of $U(\mathbf{q})$ is a immediate consequence of the fact that the matrix

$$\left.\frac{\partial^2 U(\mathbf{q})}{\partial q_\mu\,\partial q_\nu}\right|_{\bar{q}}$$

is positive definite.

One can also use a second function $V(\mathbf{p}, \mathbf{q})$ of $2n$ variables $(\mathbf{p}, \mathbf{q}) = (p_1 ..., p_n, q_1 ,..., q_n)$ from which $U(\mathbf{q})$ can be obtained by a simple replacement. The calculations done with this function are simpler in the applications and we shall also call it the effective potential. It is also represented by a series in η, $V = V_0 + \eta V_1 + \eta^2 V_2 + \cdots$, and has the property that the n equations

$$\left.\frac{\partial V(\mathbf{p}, \mathbf{q})}{\partial p_\alpha}\right|_{\mathbf{p}=\mathbf{0}} \equiv F_\alpha(\mathbf{q}) = 0 \tag{9}$$

determine the mean values in the possible metastable state as follows: $F_\alpha(\mathbf{b}) = 0$ implies $G_\alpha(\mathbf{b}) = 0$ [see (3)] and if $\mathbf{b}$ is a local minimum of $U(\mathbf{q})$, then it is a mean value in a possible metastable state in the sense explained above. This function is also defined as a Legendre transform. Consider the process defined by (1) after replacing $B_\mu(\mathbf{q}) \to B_\mu(\mathbf{q}) - J_\mu^*$, where J_μ^*, where J_μ^* is a real constant vector and let $\hat{Z}(\mathbf{J}, \mathbf{J}^*; \tau)$ be the generating functional for this new process defined by

$$\tilde{Z}(\mathbf{J}, \mathbf{J}^*; \tau) = \left\langle \exp\left(\frac{J_\mu}{\eta}\int_{t_0}^{T} dt\, q_\mu(t)\right)\right\rangle^{J^*}_{(\tau, \alpha_0)} \tag{10}$$

with the same notation as befoe and now $\langle\cdots\rangle^{J^*}$ stands for the average with respect to the new process. We write again $\tilde{Z}(\mathbf{J}, \mathbf{J}^*; \tau) = \exp[(1/\eta)\, \tilde{W}(\mathbf{J}, \mathbf{J}^*; \tau)]$ and with the same assumption concerning the limit $\tau \to \infty$ [see (5)] we have that $\tilde{W}(\mathbf{J}, \mathbf{J}^*; \tau) = \tau \bar{W}(\mathbf{J}, \mathbf{J}^*)$ in that limit and $V(\mathbf{p}, \mathbf{q})$ is defined as the double Legendre transform of $\bar{W}(\mathbf{J}, \mathbf{J}^*)$ with respect to $(\mathbf{J}, \mathbf{J}^*)$,

$$V(\mathbf{p}, \mathbf{q}) + \bar{W}(\mathbf{J}, \mathbf{J}^*) = \mathbf{J} \cdot \mathbf{q} + \mathbf{J}^* \cdot \mathbf{p} \tag{11}$$

$$\frac{\partial \bar{W}(\mathbf{J}, \mathbf{J}^*)}{\partial J_\alpha} = q_\alpha, \qquad \frac{\partial \bar{W}(\mathbf{J}, \mathbf{J}^*)}{\partial J_\alpha^*} = p_\alpha \tag{12}$$

We shall calculate explicitly here ${}^{(1)}V(\mathbf{p},\mathbf{q}) = V_0(\mathbf{p},\mathbf{q}) + \eta V_1(\mathbf{p},\mathbf{q})$ (we use from now on the notation ${}^{(k)}f = f_0 + \eta f_1 + \cdots + \eta^k f_k$ for the first k term of a series in η), i.e., the leading term and the first correction. For $U(\mathbf{q})$ the corresponding approximation ${}^{(1)}U(\mathbf{q}) = U_0(\mathbf{q}) + \eta U_1(\mathbf{q})$ is related to ${}^{(1)}V$ by

$$ {}^{(1)}U(\mathbf{q}) = {}^{(1)}V(p_\mu = g_{\mu\nu} B_\nu(\mathbf{q}), \mathbf{q}) \tag{13} $$

where $g_{\mu\nu}(\mathbf{q}) = \sigma_i^\mu(\mathbf{q}) = \sigma_i^\mu(\mathbf{q})\, c^{ij} \sigma_j^\nu(\mathbf{q})$. The result is

$$ V_0(\mathbf{p},\mathbf{q}) = p_\mu B_\nu(\mathbf{q}) - \tfrac{1}{2} g_{\mu\nu}(\mathbf{q})\, p_\mu p_\nu \tag{14} $$

$$ V_1(\mathbf{p},\mathbf{q}) = \tfrac{1}{2}\left(\operatorname{Tr} L(\mathbf{p},\mathbf{q}) + \sum_{j=1}^{n} \lambda_j(\mathbf{p},\mathbf{q}) \right) \tag{15} $$

where $\operatorname{Tr} L$ stands for the trace of the $(n \times n)$ matrix with elements $L_{\mu\nu} = \partial_\nu B_\mu(\mathbf{q}) - p_\alpha \partial_\nu g^{\mu\alpha}(\mathbf{q})$ and the $\{\lambda_j(\mathbf{p},\mathbf{q})\}$ are the eigenvalues with positive real part of the linear Hamiltonian system for the variables $(\mathbf{P},\mathbf{Q})$ generated by the quadratic Hamiltonian $\bar{h}(\mathbf{P},\mathbf{Q})$ given by

$$ \bar{h}(\mathbf{P},\mathbf{Q}) = -\tfrac{1}{2} g^{\mu\nu}(\mathbf{q})\, P_\mu P_\nu + L_{\mu\nu}(\mathbf{p},\mathbf{q})\, P_\mu Q_\nu + \tfrac{1}{2} U^{(1)}_{\mu\nu}(\mathbf{p},\mathbf{q})\, Q_\mu Q_\nu \tag{16} $$

$$ U^{(1)}_{\mu\nu}(\mathbf{p},\mathbf{q}) \equiv p_\alpha \partial_{\mu\nu} B_\alpha(\mathbf{q}) - \tfrac{1}{2} p_\alpha p_\beta \partial_{\mu\nu} g^{\alpha\beta}(\mathbf{q}) \tag{17} $$

In order to clarify the concepts we have introduced we give a simple example. Consider instead of (1) the one-variable system

$$ \dot{q} = B(q) + \sqrt{\eta}\, \xi(t) \tag{18} $$

with $\xi(t)$ a δ-correlated white noise of zero mean. The effective potential ${}^{(1)}V(p,q)$ is here (primes denotes derivatives with respect to the argument)

$$ {}^{(1)}V(p,q) = pB(q) - \frac{1}{2} p^2 + \frac{\eta}{2} \{ B'(q) + [B'(q)^2 + pB''(q)]^{1/2} \} \tag{19} $$

The mean value $\bar{q}$ is unique for this stochastic process and consequently Eq. (9) should have a unique solution. Let us consider the case in which the deterministic system $\dot{q} = B(q)$ associated with (18) has two stable attractors, for example, if $B(q) = \mu q - q^3$, $\mu > 0$, with stable fixed points $\pm\sqrt{\mu}$. If we construct a perturbation expansion for the mean value in the stationary state as a series in η we know[10,11] that we shall be able to construct two series $\langle q(t) \rangle^{\pm} = \pm[\sqrt{\mu} + O(\eta)]$ which correspond to two metastable states with an exponentially long escape time $\exp(b/\eta)$, $b > 0$, which tends to infinity when $\eta \to 0$. This expresses the fact that the deterministic system $\dot{q} = \mu q - q^3$ can go to $\sqrt{\mu}$ or $-\sqrt{\mu}$, depending on the initial condition. This fact should appear in the perturbative $V(p,q)$, and

Eq. (9) expanded as a series in η should have two roots corresponding to $\pm[\sqrt{\mu}+O(\eta)]$. This is easily verified from (19), since

$$\left.\frac{\partial\,{}^{(1)}V(p,q)}{\partial p}\right|_{p=0}=A(q)+\frac{\eta A''(q)}{4\,|A'(q)|}=0 \tag{20}$$

admits the two solutions $\pm[\sqrt{\mu}-\frac{3}{8}(\eta/\mu^2)+O(\eta^2)]$. The function ${}^{(1)}U(q)$ in this case is

$${}^{(1)}U(q)={}^{(1)}V(p=B(q),q)=\frac{1}{2}B(q)^2+\frac{\eta}{2}\{B'(q)+[B'(q)^2+B(q)\,B''(q)]^{1/2}\} \tag{21}$$

and the equation $\partial\,{}^{(1)}U(q)/\partial q=0$ [see (3)] will give again the previous result.[12]

One usually speaks of a noise-induced transition if, when varying the noise intensity, there is an essential change of $p_{st}(\mathbf{q})$,[13] for example, $p_{st}(\mathbf{q})$ changes from having one maximum before the transition to several maxima after. Another situation which we can call a noise-induced transition is a deterministic system with only one global asymptotically stable attractor and which acquires, when we add noise of intensity η, a stationary probability with more than one maximum. If this happens for arbitrary small η at least for some region of variation of the other parameters, we shall speak of a transition induced by weak noise, an example of which is given in Section 3. It appears then that in the two cases mentioned above the effective potential $V(\mathbf{p},\mathbf{q})$ can be a useful tool, since the solutions of Eq. (9) are in correspondence with the maxima of $p_{st}(\mathbf{q})$ [when they correspond to local minima of $U(\mathbf{q})$]. Moreover, the perturbative $V(\mathbf{p},\mathbf{q})$ can always be calculated as a series in powers of η in a systematic way and this is an an advantage with respect to the use of $p_{st}(\mathbf{q})$, where the expansion

$$p_{st}(\mathbf{q})=\exp[-1|\eta)(\phi_0(\mathbf{q})+\eta\phi_1(\mathbf{q})+\cdots)]$$

in which $\phi_0(\mathbf{q})$ is the Graham nonequilibrium potential, needs boundary conditions on $\phi_0(\mathbf{q})$. The conditions one imposes are that $\phi_0(\mathbf{q})$ have local minima in the attractors of the deterministic system, since then $p_{st}(\mathbf{q})=\exp[-(1/\eta)\,\phi_0(\mathbf{q})]$ will have there local maxima. But in the kinds of situations in which we are interested the maxima of $p_{st}(\mathbf{q})$ are created by the noise and we cannot impose the previous condition (this is the situation of the example of Section 3). Once we have determined the mean values using the effective potential, we can yse this information as boundary conditions for $\phi_0(\mathbf{q})$ and in this sense the effective potential is complementary to the nonequilibrium potential, which can then be calculated, s we shall illustrate in a fothcoming paper.

The expansion of $V(\mathbf{p}, \mathbf{q})$ in power of η will be especially relevant for a weak noise transition since this phenomenon can only arise from the higher order corrections to V_0 and this is the reason fo presenting here an explicit formula for the first $V_1(\mathbf{p}, \mathbf{q})$. In fact some reflection on the problem leads to the conclusion that a situation in which a weak noise transition can arise is when the function $B_\mu(\mathbf{q})$ in (1) is of the form $B_\mu(\mathbf{q}) = A_\mu(\mathbf{q}) + \eta a_\mu(\mathbf{q})$, where the parameters in $A_\mu(\mathbf{q})$ and $a_\mu(\mathbf{q})$ are of order one with respect to $\eta \ll 1$ [we shall write $O(1)$ in what follows], since then the term $\eta a_\mu(\mathbf{q})$ will contribute to V_1 as well as the noise term. In Section 2 we calculate up to first order in η the effective potential for a general system modeled by Eq. (1). The higher order terms can be calculated with similar techniques, which we shall present elsewhere. The construction of the effective potential done in Section 2 is presented in a different form with respect to the present section. The equivalence is proved in detail in Appendix C, where we also discuss carefully the absence of global convexity for the perturbative effective potential and the reason why this property is lost in our calculation. In Section 3 we use the expression obtained for the effective potential to study models exhibiting weak noise transitions. At the end of the section we justify our calculation with a critical discussion of its validity, which shows clearly how the weak noise transition arises and how the effective potential is modified in this situation. Appendix A is devoted to some technical problems related to the calculation of $V_1(\mathbf{p}, \mathbf{q})$, while in Appendix B we present an alternative discussion of the models of Section 3 involving an approximation to the stationary probability and which confirm the discussion based on the effective potential.

2. CALCULATION OF THE EFFECTIVE POTENTIAL

We consider a system modeled by a set of n stochastic differential equations (1) with $B_\mu(\mathbf{q}) = A_\mu(\mathbf{q}(t)) + \eta a_\mu(\mathbf{q}(t))$,

$$\dot{q}_\mu = A_\mu(\mathbf{q}(t)) + \eta a_\mu(\mathbf{q}(t)) + \sqrt{\eta}\, \sigma_j^\mu(\mathbf{q}(t))\, \xi^j(\mathbf{t}) \tag{22}$$

We interpret these equations in the Ito sense, which is no restriction, since if this is not the case one can always transform them to the form (22). We shall make our calculations using an equivalent definition of the effective potential. The equivalence with the previous definition will be discussed in Appendix C. The generating functional of correlation and response functions is given by[14] [$J_\mu(t)$ and $J_\mu^*(t)$ are real source and $\boldsymbol{\alpha}_0$ is a deterministic initial condition]

$$\tilde{Z}[J, J^*]$$
$$= \int_{\gamma(0)} \mathscr{D}q\, \mathscr{D}p$$
$$\times \exp\left\{ i \int_{t_0}^{T} dt \left[p_\mu \dot{q}_\mu(t) - \tilde{H}(\mathbf{p}(t), \mathbf{q}(t)) - \frac{i}{\eta} J_\mu(t) q_\mu + J_\mu^* p_\mu \right] \right\}$$
$$\times \delta(\mathbf{q}(t_0) - \boldsymbol{\alpha}_0) \tag{23}$$

where $\tilde{H} = p_\mu(A_\mu + \eta a_\mu) - \frac{1}{2} i\eta g^{\mu\nu} p_\mu p_\nu$, $\gamma(0)$ stands for prepoint discretization,[14] and we use square brackets in $\tilde{Z}$ [compare with formula (10)] to indicate that it is a funtional of $(\mathbf{J}(\cdot), \mathbf{J}^*(\cdot))$. We shall often omit $\gamma(0)$ in what follows since we shall almost always use this discretization, which defines (23) as the limit $N \to \infty$ of $\tilde{I}_N$,

$$\tilde{I}_N = \int \prod_{\mu=1}^{n} \prod_{i=1}^{N+1} dq_{\mu,i} \prod_{j=1}^{N+1} \frac{dp_{\mu,j}}{2\pi}$$
$$\times \exp i\varepsilon \sum_{j=1}^{N+1} \left[p_{\mu j} \frac{\Delta q_{\mu,j}}{\varepsilon} - \tilde{H}(p_j, q_{j-1}) - \frac{i}{\eta} J_\mu(t_j)\, q_{\mu,j} + J_\mu^*(t_j)\, p_{\mu,j} \right] \tag{24}$$

with $t_j = t_0 + j\varepsilon$, $t_{N+1} = T$, $\mathbf{q}_0 = \boldsymbol{\alpha}_0$, $\Delta q_{\mu,j} = q_{\mu,j} - q_{\mu,j-1}$, and identifying $q_{\mu,j} = q_\mu(t_j)$, $p_{\mu,j} = p_\mu(t_j)$, we see that the integral in the argument of the exponential in (23) has the corresponding quantity in (24) as a Riemann sum. One has $\tilde{Z}[0, J^*] = 1$ and we define

$$\prod_{j=1}^{n} \eta \frac{\delta}{\delta J_{\mu_j}(t_j)} \prod_{l=1}^{m} \frac{\delta}{\delta J_{\nu_j}^*(t_l')} \tilde{Z}[J, J^*]$$
$$\equiv G_{n,m}(\mu_1, t_1; ...; \mu_n, t_n \,|\, \nu_1, t_1'; ...; \nu_m, t_m' \,|\, J, J^*)$$
$$= \int \mathscr{D}q\ \mathscr{D}p \prod_{j=1}^{n} q_{\mu_j}(t_j) \prod_{l=1}^{m} ip_{\nu_l}(t_l')$$
$$\times \exp\left\{ i \int_{t_0}^{T} dt \left[p_\mu \dot{q}_\mu - \tilde{H} - J_\mu^* p_\mu - \frac{i}{\eta} J_\mu q_\mu \right] \right\} \cdot \delta(\mathbf{q}(t_0) - \boldsymbol{\alpha}_0) \tag{25}$$

We shall also use for these functions the notations

$$\langle q_{\mu_1}(t_1) \cdots q_{\mu_n}(t_n)\, ip_{\nu_1}(t_1') \cdots ip_{\nu_m}(t_m') \rangle^{(J, J^*)}$$

The functions $G_{n,0}(\mu_1, t_1; ...; \mu_n, t_n \,|\, 0, J^*)$ are correlation functions of the process defined by

$$\dot{q}_\mu = A_\mu(\mathbf{q}(t)) + \eta a_\mu(\mathbf{q}) - J_\mu^*(t) + \sqrt{\eta}\, \sigma_j^\mu(\mathbf{q})\ \xi^j(t) \tag{26}$$

and when $H_\mu^*(t)=0$, i.e., for the process (22), we shall simply write $\langle q_{\mu_1}(t_1)\cdots q_{\mu_n}(t_n)\rangle$ for these functions. We put

$$\tilde{Z}[J, J^*] = \exp\left(\frac{1}{\eta}\tilde{W}[J, J^*]\right),$$

$$\tilde{W}[J, J^*] = \eta \ln \tilde{Z}[J, J^*], \qquad \tilde{W}[0, J^*] = 0 \tag{27}$$

$$\frac{\delta \tilde{W}}{\delta J_\mu(t)} = \frac{1}{\tilde{Z}} G_{1,0}(\mu, t \mid J, J^*) = Q_\mu(t) \tag{28a}$$

$$\frac{\delta \tilde{W}}{\delta J_\mu^*(t)} = \frac{1}{\tilde{Z}} G_{0,1}(\mu, t \mid J, J^*) = P_\mu(t) \tag{28b}$$

The functions $\tilde{G}_{n,m}$ are linear response functions since

$$\begin{aligned}&\tilde{G}_{n,m}(\mu_1, t_1;...; \mu_n, t_n \mid \nu_1, t'_1;...; \nu_m, t'_m \mid J, J^*)\\ &\quad = \frac{\delta^n}{\delta J_{\nu_1}^*(t'_1)\cdots\delta J_{\nu_m}^*(t'_m)} \tilde{G}_{n,o}(\mu_1, t_1...; \mu_n, t_n \mid J, J^*)\end{aligned} \tag{29}$$

For $\mathbf{J}=0$ the functions $\tilde{G}_{n,o}$ are correlation functions of the process defined by (26) and then (29) shows that $\tilde{G}_{n,m}$ gives the response of the correlations to variations of the external sources $J_\mu^*(t)$.

One easily shows that

$$G_{o,m}(\nu_1, t'_1;...; \nu_m, t'_m \mid 0, J^*) = 0 \tag{30}$$

We define the functional Legendre transform $\tilde{\Gamma}[\mathbf{P}(\cdot), \mathbf{Q}(\cdot)]$ of the functional $\tilde{W}[\mathbf{J}(\cdot), \mathbf{J}^*(\cdot)]$ by

$$\tilde{\Gamma}[\mathbf{P}, \mathbf{Q}] + \tilde{W}[\mathbf{J}, \mathbf{J}^*] = \int_{t_0}^{T} dy\, [J_\mu(t)\, Q_\mu(t) + J_\mu^*(t)\, P_\mu(t)] \tag{31}$$

together with (28a), (28b). Then

$$\frac{\delta\tilde{\Gamma}[\mathbf{P}, \mathbf{Q}]}{\delta Q_\mu(t)} = J_\mu(t), \qquad \frac{\delta\tilde{\Gamma}[\mathbf{P}, \mathbf{Q}]}{\delta P_\mu(t)} = J_\mu^*(t) \tag{32}$$

When $\mathbf{J}=\mathbf{J}^*=0$ one has $\tilde{Z}[0,0]=1$, $\tilde{W}[0,0]=0$, and from (25) we have [see before (26) and (30)]

$$\left.\frac{\delta\tilde{W}[J, J^*]}{\delta J_\mu(t)}\right|_{J=J^*=0} = \langle q_\mu(t)\rangle, \qquad \left.\frac{\delta\tilde{W}}{\delta J_\mu^*(t)}\right|_{J=J^*=0} = 0 \tag{33}$$

where $\langle q_\mu(t)\rangle$ is the mean value of the process defined by (22). From (31) we obtain $\tilde{\Gamma}[\mathbf{P}, \mathbf{Q}]|_0 = 0$, where the notation means that we evaluate at $\mathbf{P}(\cdot) = \mathrm{p}$, $\mathbf{Q}(t) = \langle \mathbf{q}(t)\rangle$. We can then expand the functional $\tilde{\Gamma}$ in a Taylor series around $\mathbf{P}(\cdot) = 0$, $\mathbf{Q}(\cdot) = \langle \mathbf{q}(\cdot)\rangle$. This expansion starts with quadratic terms since $\tilde{\Gamma}|_0 = 0$ and from (28) we see that

$$\left.\frac{\delta\tilde{\Gamma}}{\delta Q_\mu(t)}\right|_0 = \left.\frac{\delta\tilde{\Gamma}}{\delta P_\mu(t)}\right|_0 = 0$$

($\mathbf{J} = \mathbf{J}^* = 0$ corresponds to $\mathbf{P}(\cdot) = 0$, $\mathbf{Q}(\cdot) = \langle \mathbf{q}(t)\rangle$ in the Legendre transform]. We have

$$\begin{aligned} &\tilde{\Gamma}[\mathbf{P}(\cdot), \mathbf{Q}(\cdot)] \\ &\quad= \sum_{n+m\geqslant 2} \frac{1}{n!\, m!} \int_{t_0}^{T} \prod_{i=1}^{n} dt_i \prod_{j=1}^{m} dt'_j \\ &\qquad\times \tilde{\Gamma}_{n,m}(\mu_1, t_1; ...; \mu_m, t_n \mid \nu_1, t'_1 ...; \nu_m, t'_m) \\ &\qquad\times \prod_{i=1}^{n} [Q_{\mu_i}(t_i) - \langle q_{\mu_i}(t_i)\rangle] \prod_{j=1}^{m} P_{\nu_j}(t'_j) \end{aligned} \tag{34a}$$

$$\begin{aligned} &\tilde{\Gamma}_{n,m}(\mu_1, t_1 ...; \mu_n, t_n \mid \nu_1, t'_1; ...; \nu_m, t'_m) \\ &\quad= \prod_{i=1}^{n} \frac{\delta}{\delta Q_{\mu_i}(t_i)} \prod_{j=1}^{m} \frac{\delta}{\delta P_{\nu_j}(t'_j)} \tilde{\Gamma}[\mathbf{P}, \mathbf{Q}]\bigg|_0 \end{aligned} \tag{34b}$$

From (30) for $J^* = 0$ we obtain that $\tilde{\Gamma}_{n,o} = 0$ and this property implies that

$$\left.\frac{\delta\tilde{\Gamma}}{\delta Q_\mu(t)}\right|_{\mathbf{P}=0} = 0$$

identically for any $\mathbf{Q}(\cdot)$, while

$$\left.\frac{\delta\tilde{\Gamma}[\mathbf{P}(\cdot), \mathbf{Q}(\cdot)]}{\delta P_\mu(t)}\right|_{\mathbf{P}=0} = 0 \tag{35}$$

provides an equation for the mean value $Q(t) = \langle \mathbf{q}(t)\rangle$ of the stochastic process defined by (22). Our interest is now in this process in the stationary state whose generating functional $Z[J, J^*]$ is obtained from (23) taking the limit $t_0 \to -\infty$. In this limit $Z[J, J^*]$ becomes independent of the initial condition $\boldsymbol{\alpha}_0$ and we define the corresponding functions $W[J, J^*]$ and $\Gamma[\mathbf{P}, \mathbf{Q}]$ in the stationary state by (27) and (30) with $\tilde{Z}$ replaced by Z. From now on in all formulas concerning the stationary state the limit

$t_0 \to -\infty$ should be understood even if it is not explicitly indicated. It can be shown that the functional $\Gamma[\mathbf{P}, \mathbf{Q}]$ has the expansion[15]

$$\Gamma[\mathbf{P}(\cdot), \mathbf{Q}(\cdot)] = \int_{t_0}^{T} dt\, [V(\mathbf{P}(t), \mathbf{Q}(t)) + \dot{P}_\mu(t)\, B_\mu(\mathbf{P}(t), \mathbf{Q}(t))$$
$$+ \dot{Q}_\mu(t)\, C_\mu(\mathbf{P}(t), \mathbf{Q}(t))$$
$$+ \text{terms depending on higher temporal derivatives}] \quad (36)$$

This expression defines the odinary function $V(\mathbf{P}, \mathbf{Q})$ which we call the effective potential. In the stationary state the functions $G_{n,m}$ and $\Gamma_{n,m}$ [see (29) and (34)] depend only on time differences $(\tau_i - \tau_j)$, where τ_l stands for t_i or t_j'. Concerning notations, when we use the symbol $\sim$ we refer to the general process defined by (22) with t_0 finite and when we omit it we are considering the stationary state of the process as explained above.

We shall obtain now an expansion of $\tilde{\Gamma}$ defined by (31) in powers of η. From (23) and (27) we can show that $\tilde{W}$ has an expansion[10, 11] $\tilde{W} = \tilde{W}_0 + \eta \tilde{W}_1 + \cdots$, which implies $\tilde{\Gamma} = \Gamma_0 + \eta \tilde{\Gamma}_1 + \cdots$. In order to see this we make in (23) and (24) the change of variables $\mathbf{p}_j \to \mathbf{p}_j/\eta$ to obtain

$$\hat{Z}[J, J^*] = \int \mathscr{D}q\, \mathscr{D}^\eta p \exp \frac{i}{\eta} \int_{t_0}^{T} dt\, [p_\mu \dot{q}_\mu - H(\mathbf{p}, \mathbf{q}) - iJ_\mu q_\mu + J_\mu^* p_\mu] \quad (37)$$

with discretized form $(\mathbf{q}_0 = \boldsymbol{\alpha}_0)$

$$\tilde{I}_N = \int \prod_{\mu=1}^{n} \prod_{i=1}^{N+1} dq_{i,\mu} \prod_{j=1}^{N+1} \frac{dp_{j,\mu}}{2\pi\eta}$$
$$\times \exp \frac{i\varepsilon}{\eta} \sum_{j=1}^{N+1} \left[p_{j,\mu} \frac{\Delta q_{j,\mu}}{\varepsilon} - H(\mathbf{p}_j, \mathbf{q}_{j-1}) - iJ_{j,\mu} q_{j,\mu} + J_{j,\mu}^* p_{j,\mu} \right] \quad (38)$$

with $H = H_0 + \eta H_1$, $H_0 = p_\mu A_\mu - \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$, $H_1 = p_\mu a_\mu$, and all the η dependence is explicitly indicated in (37), (38). We put $H_0^{(J,J^*)} = H_0 + iJ_\mu q_\mu - J_\mu^* p_\mu$ and let $p_\mu = v_\mu(t)$, $q_\mu = u_\mu(t)$ be the solutions of Hamilton equations for $H_0^{(J,J^*)}$, which are

$$\dot{u}_\mu(t) \frac{\partial H_0^{(J,J^*)}(\mathbf{v}, \mathbf{u})}{\partial v_\mu} = A_\mu(\mathbf{u}(t)) - ig^{\mu\nu}(\mathbf{u})\, v_\nu - J_\mu^*(t) \quad (39a)$$

$$\dot{v}_\mu(t) = -\frac{\partial H_0^{(J,J^*)}(\mathbf{v}, \mathbf{u})}{\partial u_\mu} = -v_\alpha \partial_\mu A_\alpha + \frac{i}{2} (\partial_\mu g^{\alpha\beta})\, v_\alpha v_\beta - iJ_\mu(t) \quad (39b)$$

with boundary conditions $\mathbf{u}(t_0)=\boldsymbol{\alpha}_0$, $\mathbf{v}(T)=0$. Here $v_\mu(t)$ is pure imaginary; putting $r_\mu = iv_\mu$, we obtain from (39) real equations. Note that $(\mathbf{u}(t), \mathbf{v}(t))$ are functionals $\mathbf{u}^{[J,J^*]}$, $\mathbf{v}^{[J,J^*]}$ of $(\mathbf{J}(t), \mathbf{J}^*(t))$ such that $\mathbf{u}=\mathbf{u}^{(0,0)}+O(\mathbf{J},\mathbf{J}^*)$, $\mathbf{v}=O(\mathbf{J},\mathbf{J}^*)$, i.e., $\mathbf{v}^{(0,0)}=0$, and $\mathbf{u}^{(0,0)}$ satisfies the deterministic equation associated with (22), $\dot{\mathbf{u}}^{(0,0)}=A(\mathbf{u}^{(0,0)}(t))$. We make in (37), (38) the change of variables $(\mathbf{p}(t), \mathbf{q}(t)) \to (\mathbf{p}'(t), \mathbf{q}'(t))$ given by

$$q_\mu(t) = u_\mu(t) + \sqrt{\eta}\, q'_\mu(t), \qquad p_\mu(t) = v_\mu(t) = v_\mu(t) + \sqrt{\eta}\, p'_\mu(t)$$

and we keep only the terms contributing to $\tilde{W}^{(1)} \equiv \tilde{W}_0 + \eta \tilde{W}_1$ which will determine $\tilde{\Gamma}^{(1)} = \tilde{\Gamma}_0 + \eta \tilde{\Gamma}_1$ (the procedure can be continued systematically to higher orders in η). We obtain [calling again $(\mathbf{p}(t), \mathbf{q}(t))$ the new variables of integration]

$$\begin{aligned}
&\hat{Z}[J, J^*] \\
&\quad = \exp\left[\frac{i}{\eta}\int_{t_0}^{T} dt\, (v_\mu \dot{u}_\mu - H_0^{(J,J^*)}(\mathbf{v}, \mathbf{u}) - \eta v_\mu a_\mu(\mathbf{u}))\right] \\
&\qquad \times \tilde{M}[\mathbf{v}(\cdot), \mathbf{u}(\cdot)]
\end{aligned} \tag{40a}$$

$$\begin{aligned}
&\tilde{M}[\mathbf{v}, \mathbf{u}] \\
&\quad = \int_{\gamma(0)} \mathscr{D}q\, \mathscr{D}p \\
&\qquad \times \exp i \int_{t_0}^{T} dt \left[p_\mu \dot{q}_\mu + \frac{i}{2} g^{\mu\nu}(\mathbf{u})\, p_\mu p_\nu + \frac{i}{2} U^{(1)}_{\mu\nu}(\mathbf{v}, \mathbf{u})\, q_\mu q_\nu - L_{\mu\nu}(\mathbf{v}, \mathbf{u})\, p_\mu q_\nu\right] \\
&\qquad \times \delta(\mathbf{q}(t_0))
\end{aligned} \tag{40b}$$

with $[f(\mathbf{p}, \mathbf{q})|^0 \equiv f(\mathbf{v}, \mathbf{u}) = f(-i\mathbf{r}, \mathbf{u})$ for any function $f]$

$$U^{(1)}_{\mu\nu}(\mathbf{v}, \mathbf{u}) = i\frac{\partial^2 H_0}{\partial q_\mu\, \partial q_\nu}\bigg|^0 = r_\alpha \partial_{\mu\nu} A_\alpha(\mathbf{u}) - \frac{1}{2} r_\alpha r_\beta \partial_{\mu\nu} g^{\alpha\beta}(\mathbf{u}) \tag{41a}$$

$$L_{\mu\nu}(\mathbf{v}, \mathbf{u}) = \frac{\partial^2 H_0}{\partial p_\mu\, \partial q_\nu}\bigg|^0 = \partial_\nu A_\mu(\mathbf{u}) - r_\alpha \partial_\nu g^{\mu\alpha}(\mathbf{u}) \tag{41b}$$

Then

$$\tilde{W}_0[J, J^*] = i\int_{t_0}^{T} dt\, [v_\mu \dot{u}_\mu - H_0^{(J,J^*)}(\mathbf{v}, \mathbf{u})] \tag{42a}$$

$$\tilde{W}_1[J, J^*] = -i\int_{t_0}^{T} dt\, v_\mu a_\mu(\mathbf{u}) + \ln \tilde{M}[\mathbf{v}, \mathbf{u}] \tag{42b}$$

and $\tilde{\Gamma}_0$ is determined by $\tilde{W}_0$ through formula (1), where $(\mathbf{J}, \mathbf{J}^*)$ have to be expressed as functionals of $(\mathbf{P}, \mathbf{Q})$ through (10) with $\tilde{W}$ replaced by $\tilde{W}_0$.

Using (39), one finds

$$\frac{\delta \tilde{W}_0[J, J^*]}{\delta J_\mu(\tau)} = u_\mu(\tau), \qquad \frac{\delta \tilde{W}_0[J, J^*]}{\delta J_\mu^*(\tau)} = iv_\mu(\tau) = r_\mu(\tau) \tag{43}$$

Comparing with (28), we see that at lowest order $Q_\mu(t) = u_\mu(t)$, $P_\mu(t) = iv_\mu(t)$, and these are the funtional relations of the Legendre transform between $(\mathbf{J}, \mathbf{J}^*)$ and $(\mathbf{P}, \mathbf{Q})$, since $(\mathbf{u}(\tau), \mathbf{v}(\tau))$ are explicit functionals [see (39)] of $(\mathbf{J}, \mathbf{J}^*)$. Using this, we obtain

$$\tilde{\Gamma}_0[\mathbf{P}(\cdot), \mathbf{Q}(\cdot)] = -\int_{t_0}^T dt\, [P_\mu \dot{Q}_\mu - P_\mu A_\mu(\mathbf{Q}) + \tfrac{1}{2} g^{\mu\nu}(\mathbf{Q})\, P_\mu P_\nu] \tag{44}$$

In the next oder $\tilde{\Gamma}^{(1)} = \tilde{\Gamma}_0 + \eta \tilde{\Gamma}_1$ is given by the Legendre transform of $\tilde{W}^{(1)} = \tilde{W}_0 + \eta \tilde{W}_1$ as

$$\tilde{\Gamma}^{(1)}[\mathbf{P}(\cdot), \mathbf{Q}(\cdot)] = -\tilde{W}^{(1)}[\mathbf{J}, \mathbf{J}^*] + \int_{t_0}^T dt\, [J_\mu Q_\mu + J_\mu^* P_\mu] \tag{45}$$

In this order we can still replace $\mathbf{u}(t) = \mathbf{Q}(t)$ and $\mathbf{v}(t) = -i\mathbf{P}(t)$ in the right-hand side of (45) (see, for example, ref. 16) and this gives, using (42),

$$\tilde{\Gamma}_1[\mathbf{P}(\cdot), \mathbf{Q}(\cdot)] = i \int_{t_0}^T dt\, v_\mu a_\mu(\mathbf{u}(t)) - \ln \tilde{M}(\mathbf{v}(\cdot), \mathbf{u}(\cdot)] \tag{46}$$

with $\tilde{M}$ given by (40b). We change in this last formula the $\gamma(0)$ discretization to the midpoint $\gamma(\frac{1}{2})$,[14, 17] to obtain $(\mathrm{Tr}\, L \equiv \sum_\mu L\mu\mu)$

$$\tilde{M}[\mathbf{v}, \mathbf{u}] = \exp\left[-\frac{1}{2} \int_{t_0}^T dt\, \mathrm{Tr}\, L(\mathbf{v}, \mathbf{u}) \right] \cdot \tilde{K}[\mathbf{v}(\cdot), \mathbf{u}(\cdot)] \tag{47a}$$

$$\tilde{K}[\mathbf{v}(\cdot), \mathbf{u}(\cdot)] = \int_{\gamma(1/2)} \mathscr{D}q\, \mathscr{D}p$$
$$\times \exp i \int_{t_0}^T dt\, [p_\mu \dot{q}_\mu - h(\mathbf{p}(t), \mathbf{q}(t), t)] \cdot \delta(\mathbf{q}(t_0)) \tag{47b}$$

$$h(\mathbf{p}, \mathbf{q}, t) = -\frac{i}{2} g^{\mu\nu}(\mathbf{u}(t))\, p_\mu p_\nu - \frac{i}{2} U_{\mu\nu}^{(1)}(\mathbf{v}, \mathbf{u})\, q_\mu q_\nu + L_{\mu\nu}(\mathbf{v}, \mathbf{u})\, p_\mu q_\nu \tag{47c}$$

where the explicit time dependence of h comes through $(\mathbf{u}(t), \mathbf{v}(t))$. The functional integral (47b) can be calculated as an implicit functional of

$(\mathbf{u}(t), \mathbf{v}(t))$ (see ref. 14, Chapter IX), but here we are only interested in the stationary state ($t_0 \to -\infty$) and in the effective potential defined by (36), which can be obtained putting $\mathbf{P}(\cdot) = \mathbf{P} = \text{const}$, $\mathbf{Q}(\cdot) = \mathbf{Q} = \text{const}$ in Γ, in which case (we omit $\sim$ since we consider the stationary state)

$$\Gamma[\mathbf{P}, \mathbf{Q}] = \tau V(\mathbf{P}, \mathbf{Q}), \qquad \tau \equiv T - t_0 \to \infty \tag{48}$$

We only need then to calculate (47b), replacing in h given by (47c) $\mathbf{u}(t) = \mathbf{Q} = \text{const}$, $\mathbf{v}(t) = -i\mathbf{P} = \text{const}$.

The value of the Gaussian integral (47b) can then be shown to be, in the limit $\tau \equiv T - t_0 \to \infty$ (see Appendix A),

$$K[\mathbf{v}(\cdot) = -i\mathbf{P}, \mathbf{u}(\cdot) = \mathbf{Q}] = \exp\left(-\frac{\tau}{2} \sum_{j=1}^{n} \lambda_j\right) \tag{49}$$

where $(\lambda_1, \lambda_2, ..., \lambda_n)$ are the eigenvalues with positive real parts of the $(2n \times 2n)$ matrix R with real elements $(1 \leqslant \mu,\ \nu \leqslant n)$

$$R_{\mu\nu} = 0; \qquad R_{\mu, n+\nu} = \delta_{\mu\nu}; \qquad R_{n+\mu, \nu} = g^{\mu\alpha} U_{\alpha\nu}; \qquad R_{n+\mu, n+\nu} = 2q^{\mu\alpha} \tilde{L}^{(a)}_{\alpha\nu} \tag{50a}$$

$$U_{\mu\nu} = U^{(1)}_{\mu\nu} + (L^T G^{-1} L)_{\mu\nu}, \qquad \tilde{L}_{\mu\nu} = (G^{-1} L)_{\mu\nu}, \qquad \tilde{L}^{(a)}_{\mu\nu} = \tfrac{1}{2}(\tilde{L}_{\mu\nu} - \tilde{L}_{\nu\mu}) \tag{50b}$$

where L^T is the transposed matrix of L, G the matrix with elements $g^{\mu\nu}$, and G^{-1} its inverse. The matrix R corresponds to a quadratic Hamiltonian and consequently its $2n$ roots are $\{\pm\lambda_j,\ j = 1, ..., n\}$, where we repeat the root if the multiplicity is greater than one. Moreover, R is real and this implies that the sum $\sum \lambda_j$ is real. From (45), (46), (48) we obtain in the stationary state for constants $\mathbf{P}(\cdot) = \mathbf{P}$, $\mathbf{Q}(\cdot) = \mathbf{Q}$ the result

$$\Gamma_1[\mathbf{P}, \mathbf{Q}] = \tau \left\{ P_\mu a_\mu(\mathbf{Q}) + \tfrac{1}{2} \left[\operatorname{Tr} L(-i\mathbf{P}, \mathbf{Q}) + \sum_{j=1}^{n} \lambda_j(\mathbf{P}, \mathbf{Q}) \right] \right\} \tag{51}$$

where the eigenvalues λ_j depend on $(\mathbf{P}, \mathbf{Q})$ through the dependence of the matrix R on $(\mathbf{v}(\cdot) = -i\mathbf{P},\ \mathbf{u}(\cdot) = \mathbf{Q})$. We can now write down the final result up to $O(\eta)$ for the effective potential $V(\mathbf{P}, \mathbf{Q})$ defined by (36) using (42) (we put there constants $\mathbf{P}$ and $\mathbf{Q}$), (49), and (51). We obtain $V^{(1)}(\mathbf{P}, \mathbf{Q}) = V_0 + \eta V_1$, with

$$V_0(\mathbf{P}, \mathbf{Q}) = P_\mu A_\mu(\mathbf{Q}) - \tfrac{1}{2} g^{\mu\nu}(\mathbf{Q})\, P_\mu P_\nu \tag{52a}$$

$$V_1(\mathbf{P}, \mathbf{Q}) = P_\mu a_\mu(\mathbf{Q}) + \tfrac{1}{2} \left(\operatorname{Tr} L + \sum_{j=1}^{n} \lambda_j \right) \tag{52b}$$

which are formulas (14) and (15) of Section 1.

3. MODELS PRESENTING A WEAK NOISE TRANSITION

We use here the ideas of the previous section to study a simple class of systems which exhibit, as we shall see, transitions induced by weak noise[13] which are a consequence of an extreme sensibility to noise of the models. We consider as a prototype model a deterministic system of two variables (q_1, q_2) invariant under the transformations $q_1 \to -q_1$ and $q_2 \to -q_2$ separately,

$$\dot{q}_1 = -\gamma q_1 - \alpha q_2^2 q_1 - \nu q_1^3 \equiv B_1(q_1, q_2) \tag{53a}$$

$$\dot{q}_2 = -\mu q_2 - \sigma q_2^3 + \delta q_1^2 q_2 \equiv B_2(q_1, q_2) \tag{53b}$$

where $(\alpha, \mu, \sigma, \gamma, \nu, \delta)$ are positive constants. This system has an unique attractor $(q_1 = 0,\ q_2 = 0)$ and all solutions $q_\mu(t) \to 0$, $t \to \infty$, $\mu = 1, 2$.

We add noise to this system, putting

$$\dot{q}_1 = B_1(q_1, q_2) + (\eta\sigma_1)^{1/2}\, \xi_1(t) \tag{54a}$$

$$\dot{q}_2 = B_2(q_1, q_2) + (\eta\sigma_2)^{1/2}\, \xi_2(t) \tag{54b}$$

where (ξ_1, ξ_2) are independent white noises with mean and δ-correlated, $\langle \xi_\mu(t')\, \xi_\nu(t'') \rangle = \delta_{\mu\nu}\, \delta(t' - t'')$, $\eta \ll 1$ is a small parameter measuring the intensity of the noise, and (σ_1, σ_2) are positive constants. If all the parameters involved are of $O(1)$, the effective potential is

$$V(\mathbf{p}, \mathbf{q}) = \sum_{\mu=1}^{2} \left[{}_\mu B_\mu(\mathbf{q}) - \tfrac{1}{2}\sigma_\mu p_\mu \right] + O(\eta) \tag{55}$$

and Eqs. (9) show that the mean values are $\bar{q}_\mu = 0$ (there can be no corrections of higher orders in η, due to the symmetry $q_\mu \to -q_\mu$ of the problem). If we look at Eqs. (54), we see that an interesting situation can arise if the parameters μ and σ are of $O(\eta)$. Putting $\mu = \eta\lambda$, $\sigma = \eta\beta$, $a(\mathbf{q}) = -(\lambda q_2 + \beta q_2^3)$, we can write Eqs. (54) in the form (we put $\delta = 1$ in B_2 without loss of generality)

$$\dot{q}_1 = A_1(\mathbf{q}) + (\eta\sigma_1)^{1/2}\, \xi_1(t) \tag{56a}$$

$$\dot{q}_2 = A_2(\mathbf{q}) + \eta a(\mathbf{q}) + (\eta\sigma_2)^{1/2}\, \xi_2(t) \tag{56b}$$

with $A_1 \equiv B_1(q_1, q_2)$, $A_2 = q_1^2 q_2$, and it is now necessary to calculate the effective potential up to $O(\eta)$. We obtain

$$V(\mathbf{p}, \mathbf{q}) = V_0(\mathbf{p}, \mathbf{q}) + \eta V_1(\mathbf{p}, \mathbf{q}) + O(\eta^2) \tag{57a}$$

$$V_0 = \sum_{\mu=1}^{2} \left[p_\mu A_\mu(\mathbf{q}) - \tfrac{1}{2}\sigma_\mu p_\mu \right] \tag{57b}$$

$$V_1 = p_2 a(\mathbf{q}) + \tilde{V}_1(\mathbf{p}, \mathbf{q}) \tag{57c}$$

$$\tilde{V}_1 = \tfrac{1}{2}[\mathrm{Tr}\, L + \lambda_1(\mathbf{p}, \mathbf{q}) + \lambda_2(\mathbf{p}, \mathbf{q})] \tag{57d}$$

Here $L_{\mu\nu} = \partial_\nu A_\mu(\mathbf{q})$ is independent of $\mathbf{p}$ and $(\varepsilon_1 = 1, \varepsilon_2 = -1)$

$$\lambda_j = \left(\frac{C_j}{2}\right)^{1/2}, \qquad C_j = B(\mathbf{p}, \mathbf{q}) + \varepsilon_j (B^2 - 4C)^{1/2} \tag{58a}$$

$$B(\mathbf{p}, \mathbf{q}) = \mathrm{Tr}(GU) - 4 \det(G\tilde{L}^{(a)}) \tag{58b}$$

$$C(\mathbf{p}, \mathbf{q}) = \det(GU) \tag{58c}$$

Here $G, U, L,$ and $\tilde{L}^{(a)}$ are 2×2 matrices with elements $g^{\mu\nu}$, $U_{\mu\nu} = (L^T G^{-1} L)_{\mu\nu} + U^{(1)}_{\mu\nu}$, $L_{\mu\nu}$, and $\frac{1}{2}(\tilde{L}_{\mu\nu} - \tilde{L}_{\nu\mu})$, respectively [see (50b)]. One has

$$g^{\mu\nu} = \sigma_\mu \delta_{\mu\nu}, \qquad L_{\mu\nu} = \partial_\nu A_\mu, \qquad U^{(1)}_{\mu\nu} = \sum_\alpha p_\alpha \, \partial_{\mu\nu} A_\alpha(\mathbf{q}) \tag{59}$$

We can check that ${}^{(1)}V \equiv V_0 + \eta V_1$ vanishes for $\mathbf{p} = 0$ and also that the equation

$$\left.\frac{\partial^{(1)}V}{\partial p_1}\right|_{\mathbf{p}=0} = A_1(\mathbf{q}) + O(\eta) = 0 \tag{60}$$

implies $q_1 = 0$ since $A_1(\mathbf{q})$ vanishes for $q_1 = 0$ and arbitrary q_2. According to the discussion in Section 1, [see (9)], this implies that the mean value in the stationary state is $\langle q_1 \rangle = 0$. The second equation (9), which is

$$\left.\frac{\partial^{(1)}V(\mathbf{p}, \mathbf{q})}{\partial p_2}\right|_{\mathbf{p}=0} = 0 \tag{61}$$

where we should now put $q_1 = 0$, will then determine the mean value $\langle q_2 \rangle$ in the stationary state. But here a new problem arises: one has that the eigenvalue $\lambda_2(\mathbf{p}, \mathbf{q})$ evaluated at $(\mathbf{p} = 0,\ q_1 = 0)$ vanishes for arbitrary q_2. Due to this one has simply to omit $\partial\lambda_2/\partial p_2$ in (61) [see (57)], which then reduces to

$$A_2(q_1 = 0, q_2) + \eta \left[a(q_1 = 0, q_2) + \frac{1}{2} \left.\frac{\partial \lambda_1}{\partial p_2}\right|_{\mathbf{p}=0,\, q_1=0} \right] = 0 \tag{62}$$

The reason for this prescription is that the contribution $(\lambda_1 + \lambda_2)$ to ${}^{(1)}V$ in (57) has its origin in formula (49), which is obtained by evaluating $\tilde{K}$ in

(47b) for constants ($\mathbf{v}(\cdot) = -i\mathbf{p}$, $\mathbf{u}(\cdot) = \mathbf{q}$) and in the limit $T - t_0 \equiv \tau \to \infty$. If we evaluate $\mathbf{K}$ for finite τ we obtain, as shown in Appendix A, that

$$\tilde{K}[\mathbf{v}(\cdot) = -i\mathbf{p}, \mathbf{u}(\cdot) = \mathbf{q}] = \exp[-\tfrac{1}{2}\ln F(\tau, \mathbf{p}, \mathbf{q}, \lambda_1(\mathbf{p}, \mathbf{q}), \lambda_2(\mathbf{p}, \mathbf{q}))] \quad (63)$$

where F depends on $(\mathbf{p}, \mathbf{q})$ explicitly and through $(\lambda_1(\mathbf{p}, \mathbf{q}), \lambda_2(\mathbf{p}, \mathbf{q}))$. When $\tau \to \infty$ one has $\tilde{K} \to \exp[-\frac{1}{2}\tau(\lambda_1 + \lambda_2)]$, which gives the contribution

$$\lim_{\tau \to \infty}\left(-\frac{1}{\tau}\ln \tilde{K}[\mathbf{v}(\cdot) = -i\mathbf{p}, \mathbf{u}(\cdot) = \mathbf{q}]\right) = \frac{1}{2}(\lambda_1 + \lambda_2) \quad (64)$$

to the effective potential. The function F for finite τ has the property that it is invariant under the transformations $\lambda_1 \to -\lambda_1$ and $\lambda_2 \to -\lambda_2$ separately and we can see in (64) that this property is lost in the limit, but one has to take it into account in the calculation of Eq. (61). When τ is finite the invariance $\lambda_j \to -\lambda_j$ tell us that F for small λ_2 has the form

$$F = a(\mathbf{p}, \mathbf{q}, \lambda_1, \tau) + \lambda_2^2 b(\mathbf{p}, \mathbf{q}, \lambda_1, \tau) + O(\lambda_2^4) \quad (65)$$

The contribution of $\tilde{K}$ to the derivative $\partial^{(1)}V/\partial p_2$ in (61) is

$$\lim_{\tau \to \infty}\left(-\frac{1}{\tau}\frac{\partial}{\partial p_2}\ln \tilde{K}\right) = \lim_{\tau \to \infty}\left(\frac{1}{2\tau}\frac{1}{F}\frac{\partial F}{\partial p_2}\right) \quad (66)$$

where we have to put $\mathbf{p} = 0$, $q_1 = 0$. Since $\lambda_2(\mathbf{p}, \mathbf{q})$ and $\lambda_2(\partial\lambda_2/\partial p_2)$ vanish at $\mathbf{p} = 0$, $q_1 = 0$, we see from (64) that in the calculation of $\partial F/\partial p_2$ we can put $\lambda_2 = 0$ from the beginning and then (66) gives

$$\lim_{\tau \to \infty}\left.\left(\frac{1}{2\tau}\frac{1}{F}\frac{\partial F}{\partial p_2}\right)\right|_{\mathbf{p}=0, q_1=0} = \frac{1}{2}\frac{\partial \lambda_1}{\partial p_2} \quad (67)$$

which is the prescription given after Eq. (61) and leading to (62). This prescription is general and not restricted to the case of two variables discussed here and it should be used when an eigenvalue vanishes. The reason is that it is a consequence only of the invariance of F under $\lambda_j \to -\lambda_j$. We come back now to Eq. (2), whose solutions will be the mean values $\langle q_2 \rangle^{\mathrm{st}} = \bar{q}_2$ of metastable states in the sense of Section 1 and which reduces in our model to $[A_1(q_1 = 0, q_2) = 0]$

$$\eta q_2\left[-(\lambda + \beta q_2^2) + \frac{\sigma_1}{2(\gamma + \alpha q_2^2)}\right] = 0 \quad (68)$$

This equation can predict mean values $\bar{q}_2 \neq 0$ of $O(1)$. We can understand why this happens by looking directly at Eqs. (6). Asymptotically for large times Eq. (56a) tells us that $q_1(t) \to 0$ and if we assume that $q_2(t)$ tends to a mean value $\bar{q}_2$ we can neglect in $B_1(\mathbf{q})$ the last term and then the mean value of $q_1(t)^2$ in the stationary state is

$$\langle q_1(t)^2 \rangle^{\mathrm{st}} = \frac{\eta \sigma_1}{2(\gamma + \alpha \bar{q}_2^2)}$$

In a first approximation in (56b) we can replace $q_1^2 q_2$ by $\langle q_1^2 \rangle^{\mathrm{st}} \bar{q}_2$ and $\langle q_2(t)^3 \rangle^{\mathrm{st}}$ by $\bar{q}_2^3$, which gives Eq. (68). What has happened is that on average the term $q_1^2 q_2$ dominates λq_2 and the coefficient of the linear term in q_2 becomes positive, advancing the pitchfork bifurcation arising for $\lambda < 0$. Solving (68), we obtain

$$\bar{q}_2^2 = \left[\frac{(\gamma\beta + \alpha\lambda)^2}{4\alpha^2\beta^2} + \frac{\sigma_1 - 2\lambda\gamma}{2\alpha\beta} \right]^{1/2} - \frac{\gamma\beta + \alpha\lambda}{2\alpha\beta} \tag{69a}$$

which tells that we need $\sigma_1 > 2\lambda\gamma$ in order to have bistability, i.e., two peaks in the stationary probability for q_2. We given as an illustration the numerical value of $\bar{q}_2$ for $\sigma_1 = 2$, $\lambda = 1/2$, $\alpha = \beta = \gamma = 1$; we obtain from (69) that $\bar{q}_2 = \pm 0, 52$. If we take $\alpha = 0$, we obtain now from (68) that

$$\bar{q}_2^2 = \frac{\sigma_1 - 2\lambda\gamma}{2\gamma\beta} \tag{69b}$$

and with the same values for the other parameters has now $\bar{q}_2 = \pm 1, 22$.

These values are independent of the value of $\eta \ll 1$ and we see then that the system (53a), (53b) in the region of parameters $\mu = O(\eta)$, $\sigma = O(\eta)$ exhibits a noise-induced transition. The fluctuation around the metastable states $\bar{q}_2$ given by (9) will be controlled by σ_2 in (54b) (see Appendix B) as well as the escape time, which is of the form $\exp(a/\sigma_2)$, with $a > 0$ of $O(1)$. We have done numerical simulations of the models presented here and of variations of these models which involve the same mechanism and produce similar transitions. These simulations confirm clearly the predictions of the theoretical analysis and go in some cases beyond the expected region of validity of the theory. Due to their special interest they will be represented in a fothcoming paper. We have also been able to calculate the stationary probability for the models of this section in a reasonable and controlled approximation.[1] We remark that if we consider the inhomogeneous problem, changing $q_2(t)$ to a field $\phi_2(\mathbf{x}, t)$ and adding a diffusion term[19-21] $\nabla^2 \phi_2$ to Eq. (56b), we can expect to have a real phase transition in the thermodynamic limit for σ_2 sufficiently small and if the dimension of $\mathbf{x}$ space is $d \geqslant 2$.[22-24]

Our original system defined by (4) can be interpreted if we decouple Eq. (54a) from (54b), putting, for example, $B_1 = -\gamma q_1$ and $\sigma_2 = 0$ as a process $q_2(t)$ in the presence of a nonlinear colored noise $q_1(t)^2$, where $q_1(t)$ is the Ornstein–Uhlenbeck process (see Appendix B for a discussion of this point). For processes with colored noise the possibility of noise-induced transitions has been considered by studying approximations to the stationary probability.[25] We remark that the appearance of the mean value $\bar{q}_2 \neq 0$ given by (69) in the model defined by Eqs. (56) is the analog of the Coleman–Weinberg effect in field theory.[15]

The results presented in this section must be analyzed now in relation to our general interpretation of the effective potential as a tool to find mean values of metastable states (see Section 1 and Appendix C) and our method of calculating this function. As we explain in Section 2 and Appendix C, the construction of the effective potential is done by performing a translation of variables [see (C6)] in the functional integral (37). This translation will be here given by (C6) with $(r_\mu^{(0)}(\mathbf{J}, \mathbf{J}^*),\ u_\mu^{(0)}(\mathbf{J}, \mathbf{J}^*))$ constant solutions of Hamilton's equations of the Hamiltonian

$$\bar{H}_0^{(\mathbf{J},\mathbf{J}^*)} = \bar{H}_0(\mathbf{p}, \mathbf{q}) - J_\mu^* p_\mu - J_\mu q_\mu \tag{70a}$$

$$\bar{H}_0(\mathbf{p}, \mathbf{q}) = p_1 A_1(\mathbf{q}) - \tfrac{1}{2}\sigma_1 p_1^2 + p_2 q_1^2 q_2 - \tfrac{1}{2}\sigma_2 p_2^2 \tag{70b}$$

with $A_1(\mathbf{q}) = B_1(\mathbf{q})$ given by (53a). We recall that $(r_\mu^{(0)}(\mathbf{0}, \mathbf{0}) = 0$ and $q_\mu = u_\mu^{(0)}(\mathbf{0}, \mathbf{0}))$ should be an attractor of the deterministic system $\dot{q}_\mu = A(\mathbf{q})$, i.e., a solution of

$$A_1(\mathbf{q}) = -\gamma q_1 - \alpha q_2^2 q_1 - v q_1^3 = 0 \tag{71a}$$

$$A_2(\mathbf{q}) = q_1^2 q_2 = 0 \tag{71b}$$

The solutions of (71) are $q_1 = 0$ and q_2 arbitrary, which means that $\dot{q}_\mu = A_\mu(\mathbf{q})$ has no attractors and we cannot continue with the procedure. The way to interpret the calculation is to keep all the parameters in $B_2(\mathbf{q})$ given by (53b) of $O(1)$, i.e., we do not make the separation $B_2(\mathbf{q}) = A_2(\mathbf{q}) + \eta a_2(\mathbf{q})$ indicated in (56b). The deterministic system $B_\mu(\mathbf{q}) = 0$ which replaces now (71) has one global attractor $(q_1 = u_1^{(0)}(0, 0) = 0,\ q_2 = u_2^{(0)}(0, 0) = 0)$ and we can make the translation. The effective potential will be

$${}^{(1)}V(\mathbf{p}, \mathbf{q}) = p_\mu B_\mu(\mathbf{q}) - \frac{1}{2}\sigma_\mu p_\mu^2 + \frac{\eta}{2}(\operatorname{Tr} \bar{L} + \bar{\lambda}_1 + \bar{\lambda}_2) \tag{72}$$

with $\bar{L}_{\mu\nu} = \partial_\nu B_\mu$ and $(\bar{\lambda}_1, \bar{\lambda}_2)$ depending on $(B_1(\mathbf{q}), B_2(\mathbf{q}))$. It is at this point that we consider the situation when the parameters μ and σ of $B_2(\mathbf{q})$

become of $O(\eta)$, putting $\mu=\eta\lambda$, $\sigma=\eta\beta$ [see before Eq. (6)]. Then $B_2(\mathbf{q})=A_2(\mathbf{q})+\eta a_2(\mathbf{q})$ and Eq. (72) becomes identical with (57) up to $O(\eta)$ because in the term of $O(\eta)$ in (72) we can replace $B_2(\mathbf{q})$ by $A_2(\mathbf{q})$ and then $(\bar{L}, \bar{\lambda}_1, \bar{\lambda}_2)$ become $(L, \lambda_1, \lambda_2)$ in (57). We see then very clearly in this discussion how the new metastable states appear in the effective potential.

APPENDIX A

We shall calculate here the functional integral (29b) for $\tilde{K}[\mathbf{v}(\cdot)=-i\mathbf{P}, \mathbf{u}(\cdot)=\mathbf{Q}]$ evaluated for constants $\mathbf{v}(t)$ and $\mathbf{u}(t)$ as indicated. Then $\tilde{K}$ becomes a function $\tilde{K}(\mathbf{P}, \mathbf{Q}, \tau\equiv T-t_0)$ given by (29b), where now h has no explicit time dependence and is given by

$$h(\mathbf{p}, \mathbf{q})=-\frac{i}{2}\, g^{\mu\nu}(\mathbf{Q})\, p_\mu p_\nu-\frac{i}{2}\, U^{(1)}_{\mu\nu}(-i\mathbf{P}, \mathbf{Q})\, q_\mu q_\nu+L_{\mu\nu}(-i\mathbf{P}, \mathbf{Q})\, p_\mu q_\nu \tag{A1}$$

Doing the Gaussian integral over $\mathscr{D}p$ in (29b), we obtain

$$\tilde{K}=\int_{\gamma(1/2)} \mathscr{D}q \exp\left(-\int_{t_0}^{T} dt\, \mathscr{L}(q, \dot{q})\right)\cdot\delta(\mathbf{q}(t_0)) \tag{A2}$$

where $\mathscr{D}q$ has the discretized from [see after Eq. (24) for the meaning of ε, N, and $q_{\mu,j}$]

$$\mathscr{D}q=\prod_{i=1}^{N+1}\frac{\prod_{\mu=1}^{n} dq_{\mu,i}}{[(2\pi\varepsilon)^n \det g^{\mu\nu}(\mathbf{Q})]^{1/2}} \tag{A3}$$

and $\mathscr{L}$ is the Lagrangian ($g_{\mu\nu}$ is the inverse matrix of $g^{\mu\nu}$ and sum over repeated indices is to be understood)

$$\mathscr{L}=\tfrac{1}{2}\, g_{\mu\nu}(\dot{q}_\mu-L_{\mu\alpha}q_\alpha)(\dot{q}_\nu-L_{\nu\beta}q_\beta)+\tfrac{1}{2}\, U^{(1)}_{\mu\nu}\, q_\mu q_\nu \tag{A4}$$

We calculate (2), putting $q_\mu(t)=\alpha_\mu(t)+q'_\mu(t)$ with $\boldsymbol{\alpha}(t)$ satisfying the Euler–Lagrange equations for $\mathscr{L}$ with boundary conditions $\alpha_\mu(t_0)=0$, $\alpha_\mu(T)=q_\mu$. With this change of integration variables (A2) becomes

$$\tilde{K}=\int d\mathbf{q} \exp\left(-\int_{t_0}^{T} dt\, \mathscr{L}(\alpha(t), \dot{\alpha}(t))\right)\cdot I(\mathbf{P}, \mathbf{Q}, \tau) \tag{A5}$$

$$I=\int \tilde{\mathscr{D}}q' \exp\left(-\int dt\, \mathscr{L}(q'(t), \dot{q}'(t))\right)\cdot\delta(\mathbf{q}'(t_0))\, \delta(\mathbf{q}'(T))-\mathbf{q}) \tag{A6}$$

where $\tilde{\mathscr{D}}q'$ stands now for

$$\tilde{\mathscr{D}}q' = \frac{1}{[(2\varepsilon)^n \det g^{\mu\nu}(\mathbf{Q})]^{1/2}} \prod_{i=1}^{N} \frac{\prod_{\mu=1}^{n} dq'_{\mu,i}}{[(2\pi\varepsilon)^n \det g^{\mu\nu}(\mathbf{Q})]^{1/2}} \tag{A7}$$

Since $\mathscr{L}$ is a quadratic Lagrangian and $(\alpha_\mu(t_0)) = 0$, $\alpha_\mu(T) = q_\mu)$ the action in (A5) is

$$\int_{t_0}^{T} dt\, \mathscr{L}(\alpha, \dot{\alpha}) = \frac{1}{2} \beta_\mu(T)\, q_\mu, \qquad \beta_\mu(t) \equiv \frac{\partial \mathscr{L}(\alpha, \dot{\alpha})}{\partial \dot{\alpha}_\mu(t)} \tag{A8}$$

One has that the conjugate momentum $\beta_\mu(T)$ will be here linear in $\mathbf{q}$ and of the form

$$\beta_\mu(T) = \tfrac{1}{2} M_{\mu\nu}(\tau)\, q_\nu \tag{A9}$$

where $M_{\mu\nu}(\tau)$ has a finite limit $M_{\mu\nu}$ when $\tau \to \infty$. Then in (A5) we can do the Gaussian integral over $d\mathbf{q}$ (I does not depend on $\mathbf{q}$) and we obtain

$$\tilde{K} = \frac{(2\pi)^{(n/2)}}{(\det M_{\mu\nu})^{1/2}}\, I(\mathbf{P}, \mathbf{Q}, \tau) \tag{A10}$$

The functional integral I in (6) is calculated in refs. 14, 18, and 26 and has the value

$$I = \frac{1}{[(2\pi)^n \det J^{\mu\nu}(t_0)]^{1/2}} \tag{A11}$$

where the $(n \times n)$ matrix $J^{\mu\nu}(t)$ is such that for fixed α the quantities $J^{\mu\alpha}(t)$ satisfy the Euler–Lagrange equations of $\mathscr{L}$ with boundary conditions $J^{\mu\alpha}(T) = 0$, $\dot{J}^{\mu\alpha}(T) = -g^{\mu\alpha}$. Put $x_\mu^{(\alpha)}(t) = J^{\mu\alpha}(t)$; $x_{\mu+n}^{(\alpha)} = \dot{J}^{\mu\alpha}(t)$; $\mu = 1,..., n$; $\alpha = 1,..., n$; one has that the n vectors $\mathbf{x}^\alpha$ with $2n$ components $\mathbf{x}^{(\alpha)} = (x_1^{(\alpha)},..., x_{(2n)}^{(\alpha)})$ satisfy the linear equations (here we shall explicitly indicate sums over indices)

$$\dot{x}_\mu^{(\alpha)} = \sum_{\nu=1}^{2n} R_{\mu\nu} x_\nu^{(\alpha)}, \qquad \mu = 1,..., 2n \tag{A12}$$

where the $(2n \times 2n)$ real matrix R is given in formula (50a) of Section 2. The eigenvalues of R are $(\lambda_1, \lambda_2,..., \lambda_n, \lambda_{n+1} = -\lambda_1,..., \lambda_{2n} = -\lambda_n)$ with $\operatorname{Re}\lambda_1 \geqslant \cdots \geqslant \operatorname{Re}\lambda_n \geqslant 0$ since the linear equations (A12) come from a Hamiltonian system. We assume that we can diagonalize R (if this is not the case, one needs some minor changes in the argument, but the final result will be the same) by a change of variables

$$x_\mu = \sum_{\nu=1}^{2n} \xi_{\mu\nu} y_\nu \tag{A13}$$

which reduces Eqs. (A12) to $\dot{y}_\mu = \lambda_\mu y_\mu$ with solution $y_\mu(t) = r_\mu^{(\alpha)} \exp[\lambda_\mu(t-t_0)]$, where $r_\mu^{(\alpha)}$ are initial conditions at $t = t_0$. One has

$$x_\mu^{(\alpha)}(t) = \sum_{\nu=1}^{2n} \xi_{\mu\nu} r_\nu^{(\alpha)} e^{\lambda_\nu(t-t_0)} \tag{A14}$$

and the unknown vectors $\mathbf{r}^\alpha = (r_1^{(\alpha)},..., r_{2n}^{(\alpha)})$ will be determined by imposing the boundary conditions $\mathbf{x}^{(\alpha)}(T) = \mathbf{S}^{(\alpha)}$, with $S_\mu^{(\alpha)} = 0$, $S_{n+\mu}^{(\alpha)} = -g^{\mu\alpha}$, $\mu = 1,..., n$. Since for α fixed the vectors $\boldsymbol{\xi}^{(\alpha)} = (\xi_{1\alpha},..., \xi_{2n,\alpha})$ are eigenvectors of R,

$$\sum_{\nu=1}^{2n} R_{\mu\nu} \xi_{\nu\alpha} = \lambda_\alpha \xi_{\mu\alpha}; \qquad \mu = 1,..., 2n; \quad \alpha = 1,..., 2n \tag{A15}$$

one can choose

$$\xi_{\mu\alpha} = f_{\mu\alpha}(\lambda_\alpha); \qquad \xi_{\mu,n+\alpha} = f_{\mu\alpha}(-\lambda_\alpha); \qquad \alpha = 1,..., n \tag{A16}$$

The equations for the vectors $\mathbf{r}^{(\alpha)}$ are ($\tau \equiv T - t_0$)

$$S_\mu^{(\alpha)} = \sum_{\nu=1}^{2n} \xi_{\mu\nu} r_\nu^{(\alpha)} e^{\lambda_\nu \tau} \tag{A17}$$

with solutions ($\mu = 1,..., n$)

$$r_\mu^{(\alpha)} = e^{-\lambda_\mu \tau} \frac{\det(\boldsymbol{\xi}^{(1)},..., \boldsymbol{\xi}^{(\mu-1)}, \mathbf{S}^{(\alpha)}, \boldsymbol{\xi}^{(\mu+1)},..., \boldsymbol{\xi}^{(n)}, \boldsymbol{\xi}^{(n+1)},..., \boldsymbol{\xi}^{(2n)})}{\det(\xi_{\mu\nu})} \tag{A18}$$

$$r_{n+\mu}^{(\alpha)} = e^{\lambda_\mu \tau} \frac{\det(\boldsymbol{\xi}^{(1)},..., \boldsymbol{\xi}^{(n)}, \boldsymbol{\xi}^{(n+1)},..., \boldsymbol{\xi}^{(n+\mu-1)}, \mathbf{S}^{(\alpha)}, \boldsymbol{\xi}^{(n+\mu+1)},..., \boldsymbol{\xi}^{(2n)})}{\det(\xi_{\mu\nu})} \tag{A19}$$

where the notation $\det(\mathbf{y}^{(1)},..., \mathbf{y}^{(2n)})$ means the determinant of the $(2n \times 2n)$ matrix whose columns are the components of the vectors $\mathbf{y}^{(\rho)} = (y_1^{(\rho)},..., y_{2n}^{(\rho)})$ and $\det(\xi_{\mu\nu}) = \det(\boldsymbol{\xi}^{(1)},..., \boldsymbol{\xi}^{(2n)})$. We see then that

$$r_{\mu+n}^{(\alpha)}(\lambda_1,..., \lambda_{\mu-1}, \lambda_\mu, \lambda_{\mu+1},..., \lambda_n) = r_\mu^{(\alpha)}(\lambda_1,..., \lambda_{\mu-1}, -\lambda_\mu, \lambda_{\mu+1},..., \lambda_n) \tag{A20}$$

where we used (A.16). Putting $r_\mu^{(\alpha)} = s_\mu^{(\alpha)} \exp(-\lambda_\mu \tau)$, $r_{n+\mu}^{(\alpha)} = s_{n+\mu}^{(\alpha)} \exp(\lambda_\mu \tau)$, $\mu = 1,..., n$, we can exhibit the τ dependence explicitly and from (A14) we obtain

$$J^{\mu\alpha}(t_0) \equiv x_\mu^{(\alpha)}(t_0) = \sum_{\nu=1}^{2n} \xi_{\mu\nu} s_\nu^{(\alpha)} e^{-\lambda_\nu \tau} \tag{A21}$$

From (A16) and (A20) we see that $J^{\mu\alpha}(t) \equiv x_\mu^{(\alpha)}(t)$ is invariant under the transformations $\lambda_\nu \to -\lambda_\nu$ for each ν separately and this is then *a fortiori* true for $J^{\mu\alpha}(t_0)$, a property used in Section 3. Using (A21), we can calculate $\det J^{\mu\nu}(t_0)$, which has the form

$$\det J^{\mu\nu}(t_0) = \sum_j B_j(\mathbf{P}, \mathbf{Q}, \{\lambda_i(\mathbf{P}, \mathbf{Q})\}) \exp \tau S_j \tag{A22}$$

where $S_j = \tilde{\lambda}_1 + \tilde{\lambda}_2 + \cdots + \tilde{\lambda}_n$ is a sum of n eigenvalues $\tilde{\lambda}_j$ each of them chosen from the set $(\lambda_1,..., \lambda_{2n})$. It is simple to see that if in S_j an eigenvalue appears two times, the corresponding coefficient B_j vanishes. We see then that when $\tau \to \infty$ the dominant term term in (A23) will have the form

$$\lim_{\tau \to \infty} \det J^{\mu\nu}(t_0) = B(\mathbf{P}, \mathbf{Q}, \{\lambda_i(\mathbf{P}, \mathbf{Q})\}) \exp\left(\tau \sum_{j=1}^{n} \lambda_j\right) \tag{A23}$$

which implies formula (27) in Section 2. The sum $\sum \lambda_j$ is always real, since if $(\lambda + i\Omega)$ is an eigenvalue of the real matrix R, then $(\lambda - i\Omega)$ is also one.

APPENDIX B

We derive here an approximate expression for the stationary probability $p_{\mathrm{st}}(q_2)$ of q_2 in the model defined by Eqs. (43). We remark that (43a) tells that $q_1(t)$ tends to zero in a time of $O(1)$ and we can neglect there the cubic terms q_1^3 and replace (43a) for times $t > O(1)$ by

$$\dot{q}_1 = -q_1(\gamma + \alpha q_2^2) + (\eta\sigma_1)^{1/2} \xi_1(t) \tag{B1}$$

The variable q_2 has slow vaariations in an interval of time of $O(1)$ [this is seen in (43b) since q_1^2 is $O(\eta)$]. We can conclude from (B1) that for times $t > O(1)$ the process $q_1(t)$ is approximately in the stationary state and that it can be considered as an Ornstein–Uhlenbeck process with correlation time $[\gamma + \alpha q_2(t)^2]^{-1}$. Then in this regime we have

$$\langle q_1(t) \rangle = 0 \qquad \text{and} \qquad \langle q_1(t)^2 \rangle = \frac{\eta\sigma_1}{2[\gamma + \alpha q_2(t)^2]}$$

We can now use this in (43b), since in a first approximation we can replace $q_1(t)^2\, q_2(t)$ by $\langle q_1(t)^2\rangle\, q_2(t)$, which gives then

$$\dot{q}_2(t)=\eta\left[\left(\frac{\sigma_1}{2[\gamma+\alpha q_2(t)^2]}-\lambda\right)q_2(t)-\beta q_2^3\right]+(\eta\sigma_2)^{1/2}\,\xi_2(t) \tag{B2}$$

with stationary probability (N is a normalization factor)

$$p_{\mathrm{st}}(q_2)=N\exp\left(-\frac{1}{\sigma_2}V(q_2)\right) \tag{B3}$$

$$V(q_2)=-2\int_0^{q_2}\left[\left(\frac{\sigma_1}{2(\gamma+\alpha q^2)}+\lambda\right)q-\beta q^3\right] \tag{B4}$$

Equation (B3) shows that the fluctuations and the escape time from a metastable state are determined by σ_2 as stated in Section 3 and the maxima of $p_{\mathrm{st}}(q_2)$ [minima of $V(q_2)$] will be determined by $V'(q_2)=0$, which coincides with Eq. (55).

We consider now the case in which we decouple Eq. (43a), putting $\alpha=\nu=0$ in $B_1(q_1, q_2)$ [see (40a)]. We have then a problem of a variable $q_2(t)$ in the presence of nonlinear colored noise. If we put $\sigma_1=\gamma^2$ the system reduces to

$$\dot{q}_1=-\gamma q_1+\gamma\sqrt{\eta}\,\xi_1(t) \tag{B5}$$

$$\dot{q}_2=-\eta(\lambda q_2+\beta q_2^3)+q_1^2 q_2+(\eta\sigma_2)^{1/2}\,\xi_2(t) \tag{B6}$$

which is of the form treated by San Miguel and Sancho[27] and also considered in ref. 9, Chapter 8 (general techniques useful fo problems with nonlinear noise can be found in ref. 28). Using the results in ref. 27, we obtain a small-η approximation for the stationary probability $p_{\mathrm{st}}(q_2)$, which is (except for a normalization factor)

$$p_{\mathrm{st}}(q_2)=\frac{1}{\sigma_2+(\eta\gamma/2)\,q_2^2}\exp\left(2\int^{q_2}dx\,\frac{(\gamma/2-\lambda)x-\beta x^3+(\eta\gamma/4)x)}{\sigma_2+(\eta\gamma/2)x^2}\right) \tag{B7}$$

The equation for the maxima of this stationary probability gives in lowest order in η an equation for the most probable values which coincides with (69b) when $\sigma_1=\gamma^2$, and $p_{\mathrm{st}}(q_2)$ then has two peaks for $\gamma/2>\lambda$, which was the conclusion obtained from the effective potential.

APPENDIX C

We discuss here the equivalence of the definition (36) of $V(\mathbf{P}, \mathbf{Q})$ with the one given in the Introduction. We recall the notation we have been using: for any quantity S depending on the source (J, J^*) and other parameters $\{\sigma\}$ we write $S[\mathbf{J}(\cdot), \mathbf{J}^*(\cdot), \{\sigma\}]$ with square brackets when the sources are time dependent and $S(\mathbf{J}, \mathbf{J}^*, \{\sigma\})$ with round brackets when they are constant vectors. The dependence in $\{\sigma\}$ will often be omitted or limited to the relevant parameters. We have that $\tilde{Z}(\mathbf{J}, \mathbf{J}^*; \tau)$ given by (10) has the functional integral representations (3) and (37) for $J_\mu(\cdot) = J_\mu = \text{const}$, $J^*_\mu(\cdot) = J^*_\mu = \text{const}$. We make again in (37) the displacement of variables $q_\mu(t) = \tilde{u}_\mu(t) + \sqrt{\eta}\, q'_\mu(t)$, $p_\mu(t) = \tilde{v}_\mu(t) + \sqrt{\eta}\, p'_\mu(t)$, but now $(p_\mu = \tilde{v}_\mu(t), q_\mu = \tilde{u}_\mu(t))$ are solutions of (39) with the same boundary conditions but with constant sources $(\mathbf{J}, \mathbf{J}^*)$. We define the real Hamiltonians

$$\bar{H}_0(\mathbf{P}, \mathbf{q}) \equiv iH_0^{(J,J^*)}(\mathbf{p} = -i\mathbf{P}, \mathbf{q}) = P_\mu A_\mu(\mathbf{q}) - \tfrac{1}{2} g^{\mu\nu}(\mathbf{q})\, P_\mu P_\nu \quad \text{(C1)}$$

$$\bar{H}_0^{(J,J^*)}(\mathbf{P}, \mathbf{q}) \equiv iH_0^{(J,J^*)}(\mathbf{p} = -i\mathbf{P}, \mathbf{q}) = \bar{H}_0(\mathbf{p}, \mathbf{q}) - J^*_\mu P_\mu - J_\mu q_\mu \quad \text{(C2)}$$

Putting $\tilde{r}_\mu(t) = i\tilde{v}_\mu(t)$, we have that $(P_\mu(t) = \tilde{r}_\mu(t; J, J^*), q_\mu(t) = \tilde{u}_\mu(t; J, J^*))$ are solutions of Hamilton's equations for $\bar{H}_0^{(J,J^*)}$, which are

$$\dot{q}_\mu = A_\mu(\mathbf{q}) - g^{\mu\nu} P_\nu - J^*_\mu \qquad \text{(C3a)}$$

$$\dot{p}_\mu = -p_\alpha \partial_\mu A_\alpha(\mathbf{q}) + \tfrac{1}{2}(\partial_\mu g^{\alpha\beta}(\mathbf{q}))\, P_\alpha P_\beta + J_\mu \qquad \text{(C3b)}$$

When $\mathbf{J} = \mathbf{J}^* = 0$ these equations admit $\mathbf{P} = 0$ as an invariant manifold and the previous solution is $(\tilde{r}_\mu(t; 0, 0) = 0, \tilde{u}_\mu(t; 0, 0))$ with $\tilde{u}_\mu(t; 0, 0)$ satisfying the deterministic equation

$$\partial_t \tilde{u}_\mu(t; 0, 0) = A_\mu(\tilde{u}_\nu(t; 0, 0)) \qquad \text{(C4)}$$

for the boundary condition $\tilde{u}_\mu(t_0; 0, 0) = \alpha_{0\mu}$. Then for each attractor $\mathbf{b}$, $A_\mu(\mathbf{b}) = 0$, of the deterministic system, and if $\boldsymbol{\alpha}_0$ is in the basin of attraction of $\mathbf{b}$, the solution $\tilde{u}_\mu(t; 0, 0)$ will be, after a transient, infinitesimally near $\mathbf{b}$ and spend there an infinite time in the limit $t_0 \to -\infty$. The duration of the transient will be determined by the eigenvalue with the smallest real part of the linearized deterministic system at $\mathbf{q} = \mathbf{b}$. At least for $(\mathbf{J}, \mathbf{J}^*)$ in a neighborhood $\tilde{\mathscr{D}}_1$ of $(\mathbf{0}, \mathbf{0})$ we shall have a constant solution $(r^{(0)}_\mu(\mathbf{J}, \mathbf{J}^*), u^{(0)}_\mu(\mathbf{J}, \mathbf{J}^*))$ for (C3) which reduces to

$$r^{(0)}_\mu(\mathbf{J} = 0, \mathbf{J}^* = 0) = 0, \qquad u^{(0)}_\mu(\mathbf{J} = 0, \mathbf{J}^* = 0) = b_\mu \qquad \text{(C5)}$$

for vanishing sources and once again the solution $(\tilde{r}^{(0)}_\mu(\mathbf{J}, \mathbf{J}^*), \tilde{u}^{(0)}_\mu(\mathbf{J}, \mathbf{J}^*))$ will spend an infinite time at the constant solution $(r^{(0)}_\mu(\mathbf{J}, \mathbf{J}^*), u^{(0)}_\mu(\mathbf{J}, \mathbf{J}^*))$ in the limit $t_0 \to -\infty$, $T \to +\infty$, i.e., $\tau \to +\infty$. The region $\tilde{\mathscr{D}}_2$ in the space $(\mathbf{p}, \mathbf{q})$ is defined by $(\mathbf{p} = r^{(0)}_\mu(\mathbf{J}, \mathbf{J}^*), \mathbf{q} = u^{(0)}_\mu(\mathbf{J}, \mathbf{J}^*))$ for $(\mathbf{J}, \mathbf{J}^*) \in \tilde{\mathscr{D}}_1$. Due to this the displacement of variables mentioned above in (37) can be effectively replaced in the calculations with a constant translation

$$p_\mu(t) = -ir_\mu^{(0)}(\mathbf{J}, \mathbf{J}^*) + \sqrt{\eta}\, p'_\mu(t), \qquad q_\mu(t) = u_\mu^{(0)}(\mathbf{J}, \mathbf{J}^*) + \sqrt{\eta}\, q'_\mu(t) \tag{C6}$$

writing

$$\tilde{Z}(\mathbf{J}, \mathbf{J}^*; \tau) = \exp\left(\frac{1}{\eta} W(\mathbf{J}, \mathbf{J}^*; \tau)\right)$$
$$\tilde{W}(\mathbf{J}, \mathbf{J}^*); \tau) = \tilde{W}_0(\mathbf{J}, \mathbf{J}^*; \tau) + \eta \tilde{W}_1(\mathbf{J}, \mathbf{J}^*; \tau)) + \cdots$$

we obtain the analog of formulas (40)–(42) with $(\mathbf{v}(\cdot), \mathbf{u}(\cdot))$ replaced by the time-independent vectors $(-i\mathbf{r}^{(0)}(\mathbf{J}, \mathbf{J}^*), \mathbf{u}^{(0)}(\mathbf{J}, \mathbf{J}^*))$. Putting $\tilde{W}(\mathbf{J}, \mathbf{J}^*; \tau) = \tau \bar{W}(\mathbf{J}, \mathbf{J}^*)$ when $\tau \to -\infty$, one has instead of (42) the formulas

$$\begin{aligned} \bar{W}_0(\mathbf{J}, \mathbf{J}^*) = &-\bar{H}_0(\mathbf{r}^{(0)}(\mathbf{J}, \mathbf{J}^*), \mathbf{u}^{(0)}(\mathbf{J}, \mathbf{J}^*)) \\ &+ J_\mu^* r_\mu^{(0)}(\mathbf{J}, \mathbf{J}^*)(\mathbf{J} + J_\mu u_\mu^{(0)}(\mathbf{J}, \mathbf{J}^*)) \end{aligned} \tag{C7}$$

$$\begin{aligned} \bar{W}_1(\mathbf{J}, \mathbf{J}^*) = &-r_\mu^{(0)}(\mathbf{J}, \mathbf{J}^*)\, a_\mu(\mathbf{u}^{(0)}(\mathbf{J}, \mathbf{J}^*)) \\ &+ \frac{1}{\tau} \ln \tilde{M}(\mathbf{r}^{(0)}(\mathbf{J}, \mathbf{J}^*), \mathbf{u}^{(0)}(\mathbf{J}, \mathbf{J}^*)) \end{aligned} \tag{C8}$$

The calculation of $C_0(\mathbf{p}, \mathbf{q})$ requires us now to take the Legendre transform of $\bar{W}_0(\mathbf{J}, \mathbf{J}^*)$. For this we have to invert

$$\frac{\partial \bar{W}_0(\mathbf{J}, \mathbf{J}^*)}{\partial J_\mu} = q_\mu, \qquad \frac{\partial \bar{W}_0(\mathbf{J}, \mathbf{J}^*)}{\partial J_\mu^*} = p_\mu \tag{C9}$$

in order to express $(\mathbf{J}, \mathbf{J}^*)$ as functions of $(\mathbf{p}, \mathbf{q})$. Equations (C9) are equivalent to saying that at order zero ($p_\mu = r_\mu^{(0)}(\mathbf{J}, \mathbf{J}^*)$, $q_\mu = u^{(0)}\mu(\mathbf{J}, \mathbf{J}^*)$), since these quantities are solution of

$$\frac{\partial \bar{H}_0(\mathbf{p}, \mathbf{q})}{\partial p_\mu} = J_\mu^*, \qquad \frac{\partial \bar{H}_0(\mathbf{p}, \mathbf{q})}{\partial q_\mu} = J_\mu \tag{C10}$$

and the inversion can be done in the region $\tilde{\mathscr{D}}_1$ and $\tilde{\mathscr{D}}_2$ introduced after (C5) with $(\mathbf{J}, \mathbf{J}^*) \in \tilde{\mathscr{D}}_1$, $(\mathbf{p}, \mathbf{q}) \in \tilde{\mathscr{D}}_2$. From $V_0(\mathbf{p}, \mathbf{q}) + \bar{W}_0(\mathbf{J}, \mathbf{J}^*) = J_\mu q_\mu + J_\mu^* p_\mu$ we obtain now using (7) and (C10) that

$$V_0(\mathbf{p}, \mathbf{q}) = \bar{H}_0(\mathbf{p}, \mathbf{q}) \tag{C11}$$

which is (52a). Instead of (5) we have now

$${}^{(1)}V_1(\mathbf{p}, \mathbf{q}) = -{}^{(1)}\bar{W}_1(\mathbf{J}, \mathbf{J}^*) + J_\mu q_\mu + J_\mu^* p_\mu \tag{C12}$$

where ${}^{(1)}\bar{W}_1 = \bar{W}_0 + \eta \bar{W}_1$ is given by (C7)–(C8) and in the right-hand side of (C12) we can still replace $\mathbf{r}^{(0)}(\mathbf{J}, \mathbf{J}^*)$ by $\mathbf{p}$ and $\mathbf{u}^{(0)}(\mathbf{J}, \mathbf{J}^*)$ by $\mathbf{q}$ [see (46)]. Since the calculation of $\tilde{M}$ is the same as in formulas (47)–(49) we formally obtain (52b) for $V_1(\mathbf{p}, \mathbf{q})$, thus showing the equivalence of the two definition of $V(\mathbf{p}, \mathbf{q})$. Notice that the eigenvalues $\{\chi_j\}$ are those of the linear system determined by the quadratic Hamiltonian $\bar{h}(\mathbf{P}, \mathbf{Q}) = ih(\mathbf{p} = -i\mathbf{P}, \mathbf{Q})$ as stated in Section 1 [see formulas (16) and (47c)] since the Lagrangian $\bar{\mathscr{L}}$ associated with $\bar{h}(\mathbf{P}, \mathbf{Q})$ is $\bar{\mathscr{L}} = -\mathscr{L}$ [see Appendix A, formula (A4)]. In order to obtain ${}^{(1)}U_1(\mathbf{q}) = U_0(\mathbf{q}) + \eta U_1(\mathbf{q})$, which is the Legendre transform of ${}^{(1)}\bar{w}(\mathbf{J}) \equiv {}^{(1)}\bar{W}(\mathbf{J}, \mathbf{J}^* = 0)$, we remark that putting $\mathbf{J}^* = 0$ in (C3) gives $\mathbf{r}^{(0)}(\mathbf{J}, \mathbf{0}) = g_{\mu\nu}(\mathbf{u}^{(0)}(\mathbf{J}, \mathbf{0}))$ for the constant solution and consequently ${}^{(1)}U(\mathbf{q}) = {}^{(1)}V(p_\mu = g_{\mu\nu}(\mathbf{q})\, A_\nu(\mathbf{q}), \mathbf{q})$, which is formula (13).

We remark here that the result for ${}^{(1)}V(\mathbf{p}, \mathbf{q})$ [formula (52)] is the same independent of the attractor $\mathbf{b}$ appearing in (C5) and in fact we do not need to know explicitly this attractor to arrive at (C2). We recall that in Section 1 we defined the function $U(\mathbf{q})$ as the Legendre transform of $\bar{w}(\mathbf{J}) \equiv \bar{W}(\mathbf{J}, \mathbf{J}^* = 0)$ and using definition (5) one can easily show that $(\partial^2 \bar{w}(\mathbf{J})/\partial J_\alpha\, \partial J_\beta)$ is a positive-definite matrix and $\bar{w}(\mathbf{J})$ a convex function. On the other hand, we can see that the inversion of $\partial \bar{w}(\mathbf{J})/\partial J_\mu = q_\mu$ up to first order in η is determined by the inversion of $\partial \bar{w}_0(\mathbf{J})/\partial J_\mu = q_\mu$. The argument is the same one just used for $\bar{W}(\mathbf{J}, \mathbf{J}^*)$ and we conclude that for $\mathbf{J}$ in a region $\mathscr{D}_1$ containing $\mathbf{J} = 0$ we can invert $q_\mu = u_\mu^{(0)}(\mathbf{J}, \mathbf{0})$ and $\mathbf{q}$ will belong to a region $\mathscr{D}_2 = \{\mathbf{q}\colon \mathbf{q} = \mathbf{u}^{(0)}(\mathbf{J}),\ \mathbf{J} \in \mathscr{D}_1\}$ containing the attractor $\mathbf{b} = \mathbf{u}^{(0)}(\mathbf{0})$. In fact, the property that the lowest order determines the inversion problem can be proved to any order in η. This procedure, which corresponds to our calculation, will then generate a function $U(\mathbf{q})$ which will be convex and have the same analytic form in each of the regions $\mathscr{D}_2$ (each one of these regions corresponds to an attractor of the deterministic system), but the global function will not be convex. A function $\bar{U}(\mathbf{q})$ which is convex everywhere can be obtained with the definition

$$\bar{U}(\mathbf{q}) = \sup_{\mathbf{J} \in R^n} (\mathbf{J} \cdot \mathbf{q} - \bar{w}(\mathbf{J})) \tag{C13}$$

This function will coincide with $U(\mathbf{q})$ in the perturbation expansion if $\bar{w}(\mathbf{J})$ is everywhere differentiable and this will not be the case for $\bar{w}_0(\mathbf{J})$ if the deterministic system has more than one attractor (an example will be given in this appendix). The function $\bar{U}(\mathbf{q})$ will be continuous, but in general not differentiable at the mean values of the metastable states and it will satisfy a large-deviation property for the random vector

$$\mathbf{y}(\tau) = \frac{1}{\tau} \int_{t_0}^{T} dt\, \mathbf{q}(t), \qquad \tau = T - t_0$$

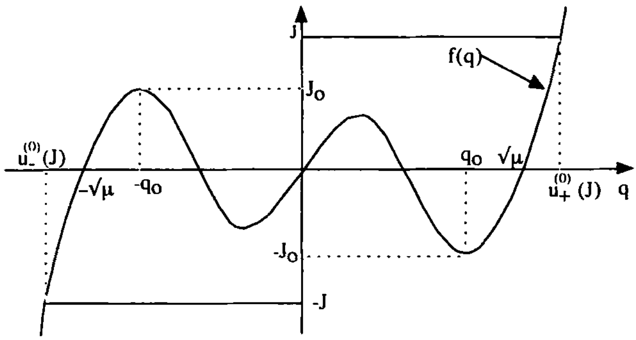

Fig. 1. Plot of the equation $f(q) = J$.

when $\tau \to \infty$.[9] Moreover, $\bar{U}(\mathbf{q})$ will take its minimal value at the true mean value of the stochastic process, which in general is different from the mean values of the metastable states. In order to illustrate this point we consider again the simple example of Section 1, Eq. (18), with $B(q) = \mu q - q^3$. For this model, Eqs. (C3), which determine the constant solution $u^{(0)}(J)$ for $J^* = 0$, become

$$f(q) \equiv B(q)\, B'(q) = J \tag{C14}$$

The function $\bar{w}_0(J)$ is here

$$\bar{w}_0(J) = -\tfrac{1}{2} B(u^{(0)}(J))^2 + J u^{(0)}(J) \tag{C15}$$

and in Fig. 1, which plots $f(q)$, we can see that $\partial \bar{w}_0(J)/\partial J = q$, which is equivalent to $q = u^{(0)}(J)$, can be inverted in two regions corresponding to the two attractors $\pm\sqrt{\mu}$.

For the attractor $\sqrt{\mu}$ we have that for $J \in \mathscr{D}_1^+ = [-J_0, \infty]$ we can solve (C14), giving $q = u_+^{(0)}(J) \in \mathscr{D}_2^+ = [q_0, \infty]$, and for $J \in \mathscr{D}_1^- = [-\infty, J_0]$ we obtain $q = u_-^{(0)}(J) \in \mathscr{D}_2^- = [-\infty, -q_0]$. One has $u_\pm^{(0)}(J=0) = \pm\sqrt{\mu}$ and $U_0(q)$ will be a convex function defined in each region $\mathscr{D}_2^\pm$ and having there the same analytic form

$$U_0(q) = \tfrac{1}{2} B(q)^2, \qquad q \in \mathscr{D}_2^\pm \tag{C16}$$

We recall that $\bar{w}(\mathbf{J})$ in (C13) is defined by (5) in terms of $\hat{Z}(\mathbf{J}; \tau)$. This function has a functional integral representation obtained from (23) for $\mathbf{J}^* = 0$ after doing there the Gaussian integral over $p_\mu(t)$. One has

$$\hat{Z}(\mathbf{J};\tau)=\int \mathscr{D}q \exp\left[-\frac{1}{\eta}\left(\frac{1}{2}\int_{t_0}^{T} dt\, g_{\mu\nu}(\dot{q}_\mu - B_\mu)(\dot{q}_\nu - B_\nu) - J_\mu \int_{t_0}^{T} dt\, q_\mu(t)\right)\right] \tag{C17}$$

$$\mathscr{D}q=\prod_{i=1}^{N+1} \frac{\prod_\mu dq_{\mu,i}}{[(2\pi\varepsilon\eta)^n \det g^{\mu\nu}(q_{i-1})]^{1/2}} \tag{C18}$$

In the simple model we are studying now we have then

$$\bar{w}(J)=\lim_{\tau\to\infty} \frac{\eta}{\tau}\hat{Z}(J,\tau) \tag{C19}$$

$$\hat{Z}(J,\tau)=\int \mathscr{D}q \exp\left(-\frac{1}{\eta}\mathscr{A}^J[q(\cdot)]\right) \tag{C20}$$

$$\mathscr{A}^J[q(\cdot)]=\frac{1}{2}\int_{t_0}^{T} dt\,(\dot{q}-B)^2 - J\int_{t_0}^{T} dt\, q(t) \tag{C21}$$

In order to generate an expansion in η for $\bar{w}(J)$ we have applied the method of steepest descent to (C19) looking for a constant solution $\tilde{q}(J)$ of the Euler–Lagrange equation of the action $\mathscr{A}^J[q(\cdot)]$ which minimizes $\mathscr{A}^J[q(\cdot)]$. In the present case we have from (C21) that

$$\bar{w}_0(J)=-\tfrac{1}{2}B(\tilde{q}(J))^2+J\tilde{q}(J) \tag{C22}$$

with

$$\tilde{q}(J)=u_+^{(0)}(J),\quad J>0,\qquad \tilde{q}(J)=u_-^{(0)}(J),\quad J<0 \tag{C23}$$

This is not what we have done in our calculation, since we have taken $\tilde{q}(J)=u_+^{(0)}(J)$ for $J\in\mathscr{D}_1^+=[-J_0,\infty]$ and $\tilde{q}(J)=u_-^{(0)}(J)$ for $J\in\mathscr{D}_1^-=[-\infty,J_0]$ (see Fig. 1) and this is why we have lost global convexity for $U_0(q)$ [see (C16)]. In fact when we choose $\tilde{q}(J)=u_+^{(0)}(J)$ for $J\in\mathscr{D}_1^+$ we have to interpret this choice as equivalent to suppressing the attractor at $(-\sqrt{\mu})$ [the same with the obvious changes with the other choice $u_-^{(0)}(J)$], but this was a good thing to do for our purpose of obtaining a differentiable $U(q)$ around the values of q corresponding to metastable states (see ref. 30 for a discussion of this point). The choice to obtain a globally convex $U(q)$ through formula (C13) is to calculate $\bar{w}_0(J)$ using (C23). In this case formula (C22) is replaced in (C13) and the calculation gives now

$$\bar{U}_0(q)=\tfrac{1}{2}B(q)^2,\qquad q\leqslant-\sqrt{\mu},\quad q\geqslant\sqrt{\mu};\qquad \bar{U}_0(q)=0,\quad q\in[-\sqrt{\mu},\sqrt{\mu}] \tag{C24}$$

This function $\bar{U}_0(q)$ is now convex everywhere, in contrast to (C16), but it is not differentiable at $q = \pm\sqrt{\mu}$. These considerations explain where convexity is lost in our calculation.

ACKNOWLEDGMENTS

The authors are grateful to Profs. G. Nicolis, D. Walgraef, and Dr. O. Descalzi for many interesting and stimulating discussions. They also acknowledge financial support from Centro de Física No Lineal y Sistemas Complejos de Santiago, Fondecyt Project 1931008/93, and CEE Project CI1*-CT92-0006.

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